Affine semigroups with maximal projective dimension

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Outline

1. Minimal free resolution of the semigroup algebra.
2. On affine semigroups with maximal projective dimension.
3. Gluing of MPD-semigroups.
4. On the irreducibility of MPD-semigroups.

Notation

$S \subset \mathbb{N}^d$ affine semigroup minimally generated by $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let $\Bbbk$ be an arbitrary field.

- *Semigroup algebra:* $\Bbbk[S] := \bigoplus_{\mathbf{a} \in S} \Bbbk \{\mathbf{a}\}$ *with* $\{\mathbf{a}\} \cdot \{\mathbf{b}\} = \{\mathbf{a} + \mathbf{b}\}$.
- *S-graded polynomial ring:* $R := \Bbbk[x_1, \ldots, x_n]$, *$S$-degree of x_i is $\mathbf{a}_i$.*

Definition

Given S-graded surjective $\Bbbk$-algebra morphism

$$\varphi_0 : R \longrightarrow \Bbbk[S];\ x_i \longmapsto \{\mathbf{a}_i\},$$

$I_S := \ker(\varphi_0)$ is the S-homogeneous binomial ideal called **ideal of S**.

Theorem

$$I_S = \left\langle \left\{ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \sum_{i=1}^{n} u_i \mathbf{a}_i = \sum_{i=1}^{n} v_i \mathbf{a}_i \right\} \right\rangle.$$

Definition

Using S-graded Nakayama's lemma recursively

$\Downarrow$ minimal free $S-$graded resolution $\Downarrow$

$$\cdots \longrightarrow R^{s_{j+1}} \xrightarrow{\varphi_{j+1}} R^{s_j} \longrightarrow \cdots \longrightarrow R^{s_2} \xrightarrow{\varphi_2} R^{s_1} \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} \mathbb{k}[S] \longrightarrow 0,$$

where, fixed $\{\mathbf{f}_1^{(j)}, \ldots, \mathbf{f}_{s_{j+1}}^{(j)}\}$ a minimal generating set for *jth*-module of syzygies $N_j := \ker(\varphi_j)$:

- $N_0 = I_S$;
- $\mathbb{k}-$algebra homomorphism $\varphi_{j+1} : R^{s_{j+1}} \longrightarrow R^{s_j}$; $\varphi_{j+1}(\mathbf{e}_i^{(j+1)}) = \mathbf{f}_i^{(j)}$.

Theorem

- *Noetherian property of $R \rightsquigarrow s_{j+1}$ is finite.*
- *Hilbert's syzygy theorem & Auslander-Buchsbaum's formula* $\rightsquigarrow s_j = 0, \forall j > p = n - \mathrm{depth}_R \mathbb{k}[S]$.
- $n - 1 \geq p$.

BRIALES, E.; CAMPILLO, A.; MARIJUÁN, C.; PISÓN, P. *Combinatorics of syzygies for semigroup algebras. Collect. Math.* **49**(2–3) (1998), 239–256

Definition

Using S-graded Nakayama's lemma recursively

$\Downarrow$ minimal free $S-$graded resolution $\Downarrow$

$$0 \longrightarrow R^{s_p} \xrightarrow{\varphi_p} R^{s_{p-1}} \longrightarrow \cdots \longrightarrow R^{s_2} \xrightarrow{\varphi_2} R^{s_1} \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} \mathbb{k}[S] \longrightarrow 0,$$

Definition

- The integer p is called the **projective dimension of** S.
- S is a **maximal projective dimension semigroup** (MPD-semigroup) if its projective dimension is $n-1$.

Problem

Which are the maximal projective dimension semigroups?

Notation

$S \subset \mathbb{N}^d$ *affine semigroup minimally generated by* $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

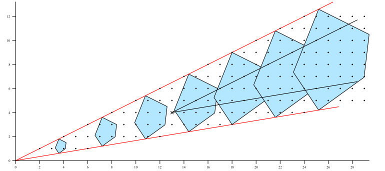
$$\mathrm{pos}(S) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a}_i \;\middle|\; \lambda_i \in \mathbb{Q}_{\geq 0},\ i = 1, \ldots, n \right\} \subset \mathbb{Q}^d_{\geq 0}$$

$\mathcal{H}(S) := (\mathrm{pos}(S) \setminus S) \cap \mathbb{N}^d$.

Definition

$\mathbf{a} \in \mathcal{H}(S)$ **pseudo-Frobenius element** of $S \leftrightsquigarrow \mathbf{a} + S \setminus \{0\} \subseteq S$.

$$\mathrm{PF}(S) := \{\mathbf{a} \in \mathcal{H}(S) \mid \mathbf{a} + S \setminus \{0\} \subseteq S\}$$



Theorem

S is a MPD-semigroup $\iff \mathrm{PF}(S) \neq \varnothing$

In this case, $\mathrm{PF}(S)$ has finite cardinality.

Corollary

Let S be a MPD-semigroup,

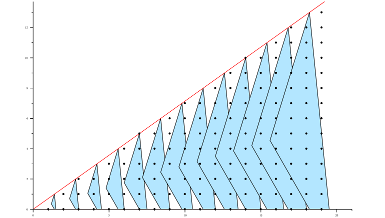
$$\mathbf{b} \in S \text{ is the } S\text{-degree of the } (n-1)th \text{ minimal syzygy of } \Bbbk[S]$$

$$\Updownarrow$$

$$\mathbf{b} \in \{\mathbf{a} + \textstyle\sum_{i=1}^{n} \mathbf{a}_i,\ \mathbf{a} \in \mathrm{PF}(S)\}.$$

Example

$S \hookrightarrow$ convex body semigroup



minimally generated by
$$\begin{pmatrix} 3 & 4 & 4 & 5 & 7 & 7 & 7 & 7 & 8 & 9 \\ 0 & 1 & 2 & 2 & 0 & 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

Computing $S-$graded minimal free resolution of $\Bbbk[S]$ using Singular:

```
LIB "toric.lib";
LIB "multigrading.lib";
ring r = 0, (x(1..10)), dp;
intmat A[2][10] =
3, 4, 4, 5, 7, 7, 7, 7, 8, 9,
0, 1, 2, 2, 0, 3, 4, 5, 1, 2;
setBaseMultigrading(A);
ideal i = toric_ideal(A,"ect");
def L = multiDegResolution(i,9,0);
```

Singular's command `multiDeg(L[9])` $\rightsquigarrow$

degrees minimal generators of $9-$th syzygy module

$$\rightsquigarrow (72, 20) \text{ and } (73, 21).$$

pseudo-Frobenius elements of S

$$\rightsquigarrow (11, 0) = (72, 20) - (61, 20)$$
$$\rightsquigarrow (12, 1) = (73, 21) - (61, 20).$$

Theorem

If $\mathbf{b} \in S$ *is an* $S-$*degree of a minimal* $j-$*syzygy of* $\Bbbk[S]$, *then* $\mathbf{b} = A\mathbf{u}$ *with* $\mathbf{u} \in \mathbb{N}^n$ *such that*

$$||\mathbf{u}||_1 \leq (1 + 4\,||A||_\infty)^{(dimrow(A))(d_j-1)} + (j+1)d_j - 1,$$

where $d_j = \left(\begin{array}{c} d \\ j+1 \end{array} \right)$.

Briales-Morales, E.; Pisón-Casares, P.; Vigneron-Tenorio, A. *The regularity of a toric variety. Journal of Algebra* **237***(1) (2001), 165–185.*

Corollary

Let S *be a MPD-semigroup. If* $\mathbf{a} \in \mathrm{PF}(S)$, *then* $\mathbf{a} = A(\mathbf{u} - \mathbf{1})$ *for some* $\mathbf{u} \in \mathbb{N}^d$ *satisfying*

$$||\mathbf{u}||_1 \leq (1 + 4\,||A||_\infty)^{(dimrow(A))(d-1)} + (d-1)d - 1.$$

Notation

Given an affine semigroup $S \subseteq \mathbb{N}^d$, denote by $G(S)$ the group spanned by S, that is,

$$G(S) = \left\{ \mathbf{a} - \mathbf{b} \in \mathbb{Z}^m \mid \mathbf{a}, \mathbf{b} \in S \right\}.$$

Definition

Let $A_1 \cup A_2 \subset \mathbb{N}^d$ be the minimal generating set of S, and S_i be the semigroup generated by A_i, $i \in \{1, 2\}$.
S is the **gluing** of S_1 and S_2 by $\mathbf{d}$ ($S = S_1 +_{\mathbf{d}} S_2$) if

- $\mathbf{d} \in S_1 \cap S_2$,
- $G(S_1) \cap G(S_2) = \mathbf{d}\mathbb{Z}$.

Theorem (Assume that $S = S_1 +_{\mathbf{d}} S_2$)

S_1 and S_2 MPD-semigroups, and $\mathbf{b}_i \in \mathrm{PF}(S_i)$, $i = 1, 2$, then

$\Rightarrow$ $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{d} \in \mathrm{PF}(S)$

$\Rightarrow$ *S is a MPD-semigroup.*

Example (Gluing MPD-semigroups)

- $S_1 = \{(x, y, z) \in \mathbb{N}^3 \mid z = 0\} \setminus \{(1, 0, 0)\}$ minimally generated by $\{(2, 0, 0), (3, 0, 0), (0, 1, 0), (1, 1, 0)\}$,
- $S_2 = \{(x, y, z) \in \mathbb{N}^3 \mid x = y\} \setminus \{(0, 0, 1)\}$ minimally generated by $\{(1, 1, 0), (1, 1, 1), (0, 0, 2), (0, 0, 3)\}$

In that case, $(1, 0, 0) \in \mathrm{PF}(S_1)$ and $(0, 0, 1) \in \mathrm{PF}(S_2)$.

$S_1 +_{(1,1,0)} S_2$ is minimally generated by
$\{(2, 0, 0), (3, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1), (0, 0, 2), (0, 0, 3)\}$

$(1, 0, 0) + (0, 0, 1) + (1, 1, 0) = (2, 1, 1)$ belongs to $\mathrm{PF}(S_1 +_{(1,1,0)} S_2)$.

Definition

Let $S \subset \mathbb{N}^d$ be a semigroup, S is **irreducible** if cannot be expressed as an intersection of two semigroups containing it properly.

Theorem

S irreducible MPD-semigroup

$$\Downarrow$$

$$\textit{either } \mathrm{PF}(S) = \{\mathbf{f}\} \textit{ or } \mathrm{PF}(S) = \{\mathbf{f}, \mathbf{f}/2\}$$

Definition

- $(\mathrm{pos}(S) \setminus S) \cap \mathbb{N}^d$ finite $\rightsquigarrow$ S is called $\mathcal{C}-$**semigroup** ($\mathcal{C} = \mathrm{pos}(S)$).
- S $\mathcal{C}-$semigroup is $\mathcal{C}-$**irreducible** if $\forall S_1$ and S_2 affine semigroups containing S with $\mathrm{pos}(S_1) = \mathrm{pos}(S_2) = \mathrm{pos}(S)$, $S \neq S_1 \cap S_2$.

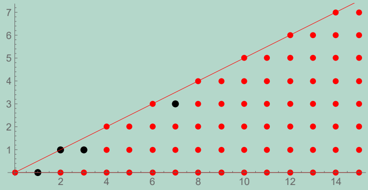
Proposition

S $\mathcal{C}-$semigroup such that $\mathrm{PF}(S) = \{\mathbf{f}\}$ *or* $\mathrm{PF}(S) = \{\mathbf{f}, \mathbf{f}/2\}$

$$\Updownarrow$$

S is $\mathcal{C}-$irreducible.

Example



Thanks for your attention!

Grupo de investigación ASCA

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