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Automaton Structures: Decision Problems and Structure Results

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with

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10th June 2019

Presentations

- traditional presentation of algebraic structures:

In this talk: semigroups, inverse semigroups or groups

$$\langle q_1, \dots, q_n \mid \ell_1 = r_1, \dots, \ell_m = r_m \rangle$$

$Q = \{q_1, \dots, q_n\}$: generators,
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- possible input to algorithms if both sets are finite
- alternative: use automata $\rightsquigarrow$ automaton structures
- Why? Many examples of groups with interesting properties arise in this way (intermediate growth, Burnside problem, ...) and allows finite description of possibly non-finitely presented (semi)groups.
- How? $\rightsquigarrow$ short recap

Automata

In this setting, an automaton is a

- possibly partial
- finite-state,
- letter-to-letter

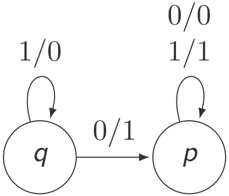
often only complete automata are considered

transducer

- without final or initial states.

A deterministic automaton is an S-automaton.

Example



S-Automata

- Idea: every state q induces a (partial) function $q \circ : \Sigma^* \to \Sigma^*$ (only requires determinism)

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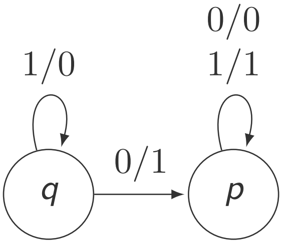

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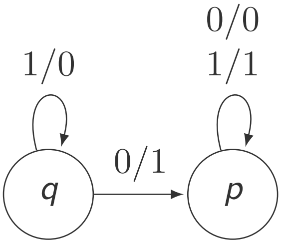

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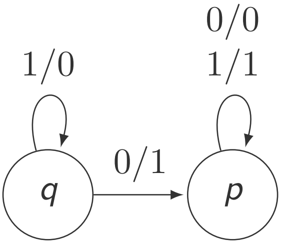

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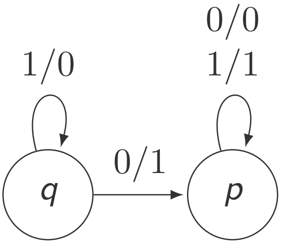

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- Notice: the functions are total if the automaton is complete.

Automaton Semigroups

- S-automaton $\mathcal{T}$ with state set Q generates semigroup: $\mathscr{S}(\mathcal{T})$ is the closure of $Q \circ = \{q \circ \mid q \in Q\}$ under composition of (partial) functions

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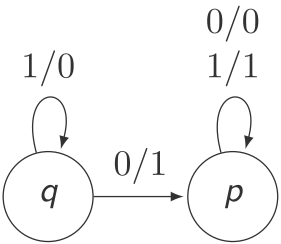

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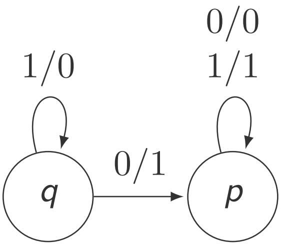

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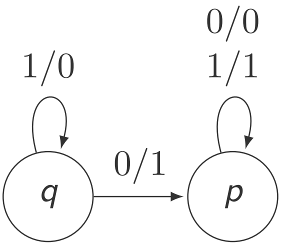

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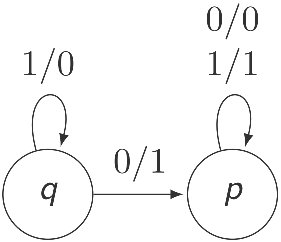

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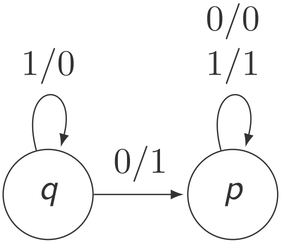

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$\mathscr{S}(\mathcal{T}) = q^*$

Inverses

- Idea: swap input and output

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- Problem: result may be not deterministic ⇝ invertible or $\bar{S}$-automaton

Overview

Other people usually simply use "automaton semigroup" for this!

automaton	properties	structure
S-automaton	deterministic	(partial) automaton semigroup
complete S-automaton	deterministic, complete	complete automaton semigroup
$\bar{S}$-automaton	deterministic, invertible	automaton-inverse semigroup
G-automaton	deterministic, complete, invertible	automaton group

This is the same as "inverse automaton semigroup" (D'Angeli, Rodaro, W. arXiv 2018).

Why bother with partial automata?

- They allow for a natural presentation of inverse automaton semigroups.
- Thus, they bridge the gap between semigroups and groups…
- …and this can help with algorithmic decision problems.
- There is no reason to only consider complete automaton semigroups.
- In most cases, results for complete automaton semigroups hold also for (partial) automaton semigroups.

Only little prior research: Olijnyk, Sushchansky and Słupik; Nekrashevych

Word problem

- The (uniform) word problem

Input: an S-automaton $\mathcal{T} = (Q, \Sigma, \delta)$, $\boldsymbol{p}, \boldsymbol{q} \in Q^+$
Question: is $\boldsymbol{p} = \boldsymbol{q}$ in $\mathscr{S}(\mathcal{T})$?

for automaton semigroups is in PSPACE
(simple guess and check algorithm, observed by Steinberg 2015).

Conjecture (Steinberg 2015)

There is an automaton group with PSPACE-complete word problem.

From inverse semigroups to groups

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- Groups: uniform word problem is NL-hard (D'Angeli, Rodaro, W. 2017)
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- But: Extension from inverse semigroups to groups is possible!

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Theorem (W., Weiß; arXiv by tomorrow)

There is an automaton group with a PSPACE-complete word problem.

- This is based on the original construction for inverse semigroups! (D'Angeli, Rodaro, W. 2017)
- New idea: use iterated commutators (similar to Barrington 1989, also: Gurevich and Mal'cev)

Compressed word problem

Definition

compressed word problem of an automaton group:

Constant: a G-automaton $\mathcal{T} = (Q, \Sigma, \delta)$

Input: a straight-line program encoding $\boldsymbol{q} \in (Q \cup \overline{Q})^*$

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Current state: Not really studied yet but...

- Problem is in ExpSpace (by uncompressing the straight-line program)
- Gillibert's result on the order problem (2018) can be adapted: there is an automaton group with a PSpace-hard compressed word problem
- Proof of D'Angeli, Rodaro, W. for word problem shows: there is an inverse automaton semigroup with an ExpSpace-complete compressed word problem.

Both: Gillibert via personal communication

ExpSpace-complete

Theorem (W., Weiß; arXiv by tomorrow)

There is an automaton group with an ExpSpace*-complete compressed word problem.*

In fact: we can use the same group for PSpace and ExpSpace!

- Hardness follows the same proof ideas as for the normal word problem.
- To the best of our knowledge: first example of a group with provably more difficult compressed word problem than (normal) word problem.

Finiteness Problem

Theorem (Gillibert, 2014)

The finiteness problem for complete automaton semigroups

Input: *a complete S-automaton* $\mathcal{T} = (Q, \Sigma, \delta)$
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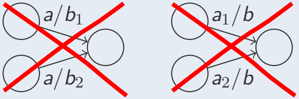

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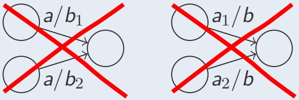

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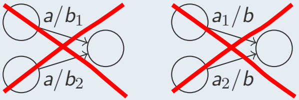

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Warning: does not show undecidability of finiteness problem for inverse automaton semigroups!

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Corollary

$\mathcal{T} = (Q, \Sigma, \delta)$: *G-automaton*

$$|\mathscr{G}(\mathcal{T})| = \infty \iff \exists \xi \in \Sigma^\omega : |\mathscr{G}(\mathcal{T}) / \operatorname{Stab}(\xi)| = \infty$$

Open Problems

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3. Do we have: S partial automaton semigroup $\iff$ S complete automaton semigroup ?
   Of course: Every complete automaton is in particular a partial one.
   Equivalent (D'Angeli, Rodaro, W. arXiv 2018) to Cain's problem (2009):

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4. Which semigroups can be presented as (complete or partial) automaton semigroups?

Thank you!