Amenability of semigroups

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- A subspace X of $\ell^\infty(S)$ is left invariant if $l_t f \in X$ for all $f \in X$ and $t \in S$.
- We can consider the right translate $r_t f$ of f and right invariant subspaces of $\ell^\infty(S)$ similarly.

The dual space of a Banach space

Let X be a Banach space. A bounded linear functional on X is a function $F: X \to \mathbb{R}$ satisfying

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \qquad (x, y \in X, \alpha, \beta \in \mathbb{R})$$

and that there is $M_F \geq 0$ such that

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X^* is indeed a Banach space with the norm $\|F\| \quad (F \in X^*)$ defined above. We call X^* the dual space of X.

Left amenable semigroups

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The semigroup S is called left amenable if there is a LIM on $\ell^\infty(S)$.

- Every commutative semigroup is left amenable.
- Every finite group is left amenable, so is any solvable group.
- The free semigroup on two generators is not left amenable, neither is the free group on two generators.

Left reversible semigroup

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Proposition 2

If S is left amenable, then it is left reversible.

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Proposition 2

If S is left amenable, then it is left reversible.

Proof.

Let m be a LIM on $\ell^\infty(S)$. If S were not left reversible, then there are $a, b \in S$ such that $aS \cap bS = \emptyset$.

Let f be the characteristic function of aS, i.e. $f(s) = 1$ for $s \in aS$ and $= 0$ for $s \notin aS$. Then $l_a f = \mathbf{1}$ and $l_b f = \mathbf{0}$. Since m is left invariant we have

$$m(f) = m(l_a f) = m(\mathbf{1}) = 1, \quad \text{and} \quad m(f) = m(l_b f) = m(\mathbf{0}) = 0.$$

This is a contradiction. □

$AP(S)$

A function $f \in \ell^\infty(S)$ is called almost periodic on S if its left orbit

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$WAP(S)$

Let X be a Banach space. The weak topology of X is the topology on X with the subbase the sets

$$\mathcal{N}_{(x,F,\varepsilon)} = \{y \in X : |F(y-x)| < \varepsilon\},$$

where $x \in X$, $F \in X^*$ and $\varepsilon > 0$.

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Denote by $WAP(S)$ the set of all weakly almost periodic functions on S. Then

- $WAP(S)$ is a left invariant closed subspace of $\ell^\infty(S)$,
- $AP(S) \subset WAP(S)$.

Open questions

The following implications are well-known.

1. If S is left amenable, then it is left reversible.
2. (R. Hsu, 1985) If S is left reversible then $WAP(S)$ has a LIM.
3. If $WAP(S)$ has a LIM, then $AP(S)$ has a LIM.

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Whether the converses of (2) and (3) hold are long-standing open problems.

Bicyclic semigroups

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We investigate partially bicyclic semigroups

$$S_2 = \langle e, a, b, c : ab = ac = e \rangle \text{ and } S_{1,1} = \langle e, a, b, c, d : ab = cd = e \rangle$$

where e denotes the unit element in each case.

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- (Mitchill, 1984) S_2 and $S_{1,1}$ are not left reversible but $AP(S_2)$ and $AP(S_{1,1})$ both have LIM.
- (Lau-Zhang) S_2 is right amenable (but not left amenable); while $S_{1,1}$ is neither left no right amenable,

where by right amenable for S we mean that there is a right invariant mean on $\ell^\infty(S)$.

Answer to the open problems

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Theorem 3 (Lau-Zhang)

- *$WAP(S_2)$ has a LIM but S_2 is not left reversible.*
- *$WAP(S_{1,1})$ has no LIM but $AP(S_{1,1})$ has a LIM.*

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- *$WAP(S_{1,1})$ has no LIM but $AP(S_{1,1})$ has a LIM.*

With Theorem 3 we have the following complete relation chains.

S is l. amen. $\Rightarrow$ S is l. rev. $\Rightarrow$ $WAP((S)$ has LIM $\Rightarrow$ $AP(S)$ has LIM

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Thank You!