Wave-shaped round functions and primitive groups

Joint work with M. Calderini, R. Civino, M. Sala and I. Zappatore[1]

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SandGAL 2019

[1] R. Aragona, M. Calderini, R. Civino, M. Sala, I. Zappatore, *Wave-Shaped Round Functions and Primitive Groups*, Advances in Mathematics of Communications 13(1), 67-88 (2019).

Block ciphers

Let $V \stackrel{\text{def}}{=} (\mathbb{F}_2)^n$ be the message space

Block cipher

A block cipher $\mathcal{C}$ is a set of (bijective) encryption functions.

$$\{\varepsilon_k\}_{k\in\mathcal{K}} \subseteq \mathrm{Sym}(V),$$

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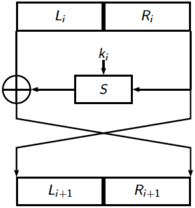
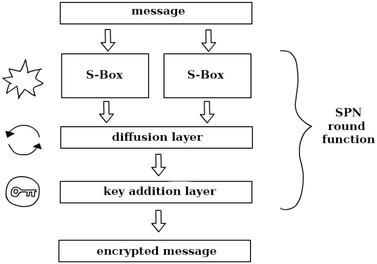
Most block ciphers are iterated block ciphers, where $\varepsilon_k = \varepsilon_{k_1} \cdots \varepsilon_{k_r}$, with $k_i \in V$, is the composition of many key-dependent permutations, known as round functions.

Iterated Block Cipher

Round of Feistel Network



Round of Translation Based (TB) Cipher
(more commonly called SPN)



Classical round function in a TB cipher

Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_b$ where each V_j is an s-dimensional brick. For each $c \in V$, the classical round function induced by c is a map $\varepsilon_c : V \to V$ where $\varepsilon_c = \gamma\lambda\sigma_c$ and

- $\gamma \in \mathrm{Sym}(V)$ is a non-linear bricklayer transformation which acts in parallel way on each V_j

  $$\gamma \qquad \boxed{\gamma'}\boxed{\gamma'}\boxed{\quad\cdots\quad}\boxed{\gamma'}$$

  The map $\gamma' : V_j \to V_j$ is traditionally called an S-box
- $\lambda \in \mathrm{Sym}(V)$ is a linear map, called mixing layer
- $\sigma_c : V \to V, x \mapsto x + c$ represents the key addition, where $+$ is the usual bitwise XOR on $\mathbb{F}_2$

Non-linearity

Let $f : (\mathbb{F}_2)^s \to (\mathbb{F}_2)^t$. Given $u \in (\mathbb{F}_2)^s$ and $v \in (\mathbb{F}_2)^t$ we define

$$\delta(f)_{u,v} \stackrel{\text{def}}{=} |\{x \in (\mathbb{F}_2)^s \mid x\hat{f}_u = xf + (x+u)f = v\}|$$

The differential uniformity of f is

$$\delta(f) \stackrel{\text{def}}{=} \max_{\substack{u \in (\mathbb{F}_2)^s, u \neq 0 \\ v \in (\mathbb{F}_2)^t}} \delta(f)_{u,v},$$

and f is said δ-differentially uniform if $\delta(f) = \delta$.

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Note that δ-differentially uniform functions with small δ are "farther" from being linear (when f is linear $\delta = 2^s$).

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Unfortunately NO APN permutation of even dimension has yet been found except one of dimension 6.

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Is it possible to define an iterated block cipher with such APN S-Boxes?

Generalisation of Round Functions: Wave functions

Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_b$,
$\dim_{\mathbb{F}_2}(V) = n$ and $\dim_{\mathbb{F}_2}(V_i) = s$.

Let $W = W_1 \oplus W_2 \oplus \ldots \oplus W_b$,
$\dim_{\mathbb{F}_2}(W) = m \geq n$ and $\dim_{\mathbb{F}_2}(W_i) = t$.

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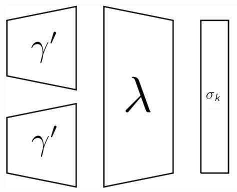
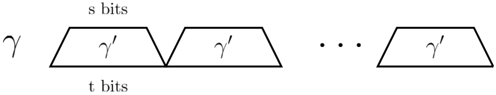


For each $c \in V$, we define a Wave function induced by c as a map $\varepsilon_c : V \to V$, where $\varepsilon_c = \gamma\lambda\sigma_c$ and

- $\gamma : V \to W$ is an injective non-linear bricklayer transformation which acts independently on each V_j



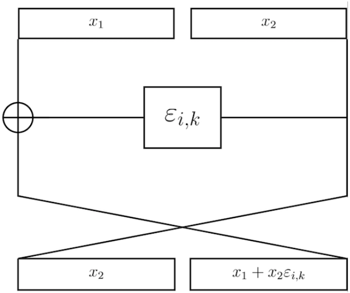
  The map $\gamma' : V_j \to W_j$ is an injective S-box
- $\lambda : W \to V$ is a surjective mixing linear
- $\sigma_c : V \to V$ is the key addition

Wave ciphers

In order to guarantee an efficient decryption, we propose to use Wave functions inside a Feistel Network

An r-round Wave block cipher $\mathcal{C}$ is a family of encryption functions $\{\varepsilon_k \mid k \in \mathcal{K}\} \subseteq \mathrm{Sym}(V \times V)$ such that for each $k \in \mathcal{K}$ $\varepsilon_k = \overline{\varepsilon_{1,k}\varepsilon_{2,k}} \ldots \overline{\varepsilon_{r,k}}$, where $\overline{\varepsilon_{i,k}}$ is the formal operator

$$\overline{\varepsilon_{i,k}} = \begin{pmatrix} 0_n & 1_n \\ 1_n & \varepsilon_{i,k} \end{pmatrix}$$



and $\varepsilon_{i,k} = \gamma\lambda\sigma_{k_i}$ is an n-bit Wave function.

Group Theoretical Security of Block ciphers

Weaknesses based on group theoretical properties

Let $\mathcal{C}$ be an r-round iterated block cipher acting on V.
The group generated by the round functions

$$\Gamma_\infty(\mathcal{C}) \stackrel{\text{def}}{=} \langle \varepsilon_{k,h} \in \mathrm{Sym}(V) \mid k \in \mathcal{K}, h = 1, \ldots, r \rangle$$

can reveal dangerous weaknesses of the cipher:

- the group is too small (Kaliski, Rivest and Sherman, 1998)
- the group is of affine type (Calderini and Sala, 2015)
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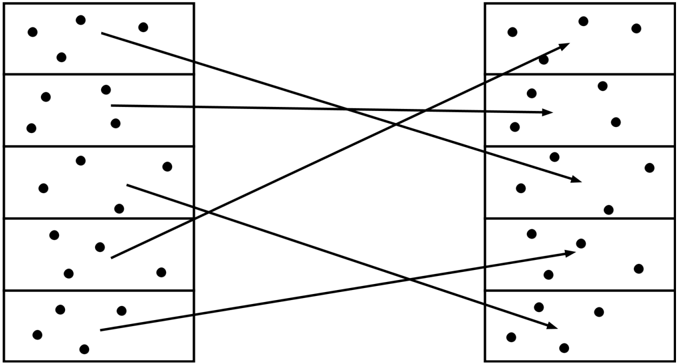
Under some cryptographic assumptions for a Wave cipher $\mathcal{C}$

$\Gamma_\infty(\mathcal{C})$ is primitive

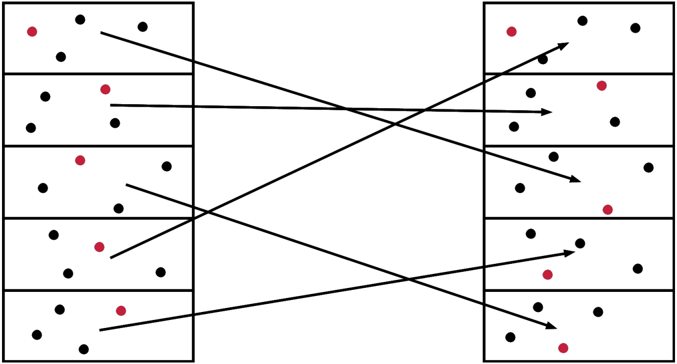
Imprimitive attack

Let $\mathcal{C}$ be an r-round iterated block cipher.
Suppose that $\Gamma_\infty(\mathcal{C})$ is imprimitive, then there exists a partition $\mathcal{B}$ of V such that for any encryption function $\varepsilon_k \in \Gamma_\infty$, we have $B\varepsilon_k \in \mathcal{B}$ for all $B \in \mathcal{B}$.



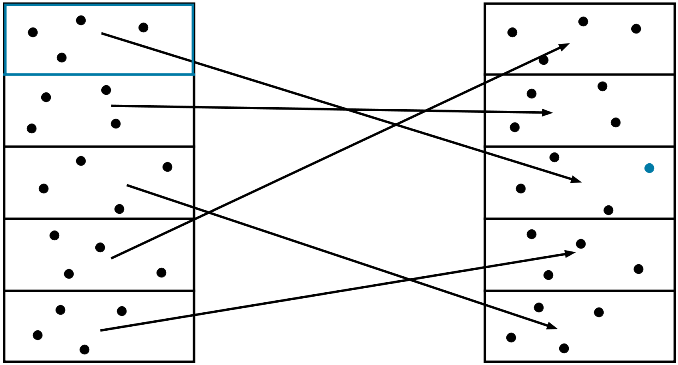
Imprimitive attack

Preprocessing performed ones per key:



Imprimitive attack

Real-time processing:



Group generated by the round functions of a Wave cipher

Let $\varepsilon_{i,k} = \gamma\lambda\sigma_{k_i} \in \mathrm{Sym}(V)$ be an n-bit Wave function. We define

$$\Gamma_i \overset{\text{def}}{=} \langle \varepsilon_{i,k} \mid k \in \mathcal{K} \rangle \quad \text{and} \quad \overline{\Gamma}_i \overset{\text{def}}{=} \langle \overline{\varepsilon_{i,k}} \mid k \in \mathcal{K} \rangle,$$

where

$$\overline{\varepsilon_{i,k}} = \begin{pmatrix} 0_n & 1_n \\ 1_n & \varepsilon_{i,k} \end{pmatrix}.$$

Hence

$$\Gamma_\infty(\mathcal{C}) \overset{\text{def}}{=} \langle \overline{\Gamma}_i \mid 1 \leq i \leq r \rangle.$$

Our goal

Show that $\Gamma_\infty(\mathcal{C})$ is primitive

Group generated by the round functions of a Wave cipher

Let us denote with

- $T_n \stackrel{\text{def}}{=} \{\sigma_k \mid x \mapsto x + k\} \leq \mathrm{Sym}(V)$
- $T_{(0,n)} \stackrel{\text{def}}{=} \{\sigma_{(0,k)} \mid (x_1, x_2) \mapsto (x_1, x_2 + k)\} \leq \mathrm{Sym}(V^2)$

Being $\rho \stackrel{\text{def}}{=} \gamma\lambda$ and denoting with $\overline{\rho}$ the formal operator $\begin{pmatrix} 0_n & 1_n \\ 1_n & \rho \end{pmatrix}$ one has

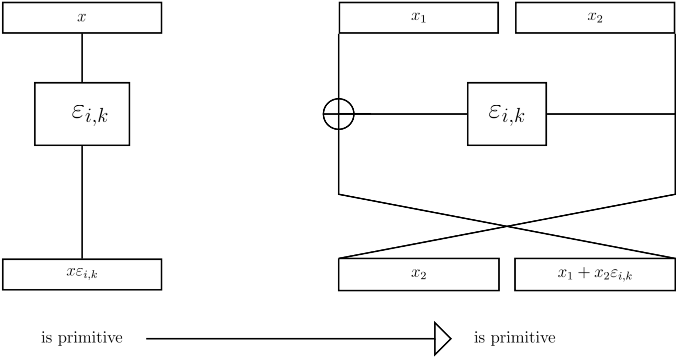
$$\Gamma_i = \langle T_n, \rho \rangle \quad \text{and} \quad \overline{\Gamma}_i(\mathcal{C}) = \langle T_{(0,n)}, \overline{\rho} \rangle,$$

for any $1 \leq i \leq r$.

Security Reduction

Theorem (A., Calderini, Civino, Sala, Zappatore)

If $\gamma\lambda \in \mathrm{Sym}(V) \setminus \mathrm{AGL}(V)$ and $\Gamma_i = \langle T_n, \rho \rangle$ is primitive on V, then $\overline{\Gamma}_i(\mathcal{C}) = \langle T_{(0,n)}, \overline{\rho} \rangle$ is primitive on $V \times V$.



Two other security assumptions for the components of a Wave function

Definition

A wall of V (resp. W) is any nontrivial sum of proper bricks of V (resp. W). A linear transformation $\lambda : W \longrightarrow V$ is a proper mixing layer if for any nontrivial wall $W' = \bigoplus_{i \in I} W_i$ of W and $V' = \bigoplus_{i \in I} V_i$ of V, where $I \subset \{1, \ldots, m\}$, then

$$V'\lambda^{-1} \not\subset W' + \operatorname{Ker} \lambda.$$

Definition

Let $j \in \{1, 2, \ldots, b\}$, $\gamma_j : V_j \to W_j$ be an S-box such that $0\gamma_j = 0$, and $\lambda : W \to V$ be a surjective linear map. Given $0 \le l < s$, γ_j is l-non-invariant with respect to λ if for any proper subspaces $V' < V_j$ and $W' < W_j$ such that $V'\gamma_j + (\operatorname{Ker} \lambda \cap W_j) = W'$, then $\dim(W') < s - l$.

Primitivity of a Wave cipher

Theorem (A., Calderini, Civino, Sala, Zappatore)

Let $\mathcal{C}$ *be Wave cipher with a proper mixing layer* λ. *If there exists* $1 \leq l < s$ *such that each S-box* γ_j *is*

- 2^l*- differentially uniform,*
- $(l-1)$*-non-invariant with respect to* λ*,*

and if $\operatorname{Ker}\lambda = \bigoplus_{j=1}^{b} \operatorname{Ker}\lambda \cap W_j$, *then* Γ_i *is primitive (and so it is* $\Gamma_\infty(\mathcal{C})$*).*

Works in progress and future works

Several problems arise from the new construction regarding Wave functions and Wave ciphers $\mathcal{C}$, such as:

- Determining conditions on the Wave functions to ensure that $\Gamma_\infty(\mathcal{C})$ is the alternating group.

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- Studying the resistance of a Wave cipher with respect to other statistical attacks, for example studying the impact of differential and linear cryptanalysis on the Wave-shaped structure.

Finally, to the best of our knowledge, $s \times t$ APN functions with $s < t$ are not very much investigated in literature.

Thanks for your attention!