SandGAL

Cremona, June 10-13, 2019

On a Graph Connecting Hyperbinary Expansions

Joint research with Maurizio Brunetti
Università di Napoli 'Federico II'

Binary Expansions

Each positive integer n can be expressed uniquely in the form

$$n = x_1 2^{k-1} + x_2 2^{k-2} + \cdots + x_{k-1} 2 + x_k.$$

where $x_i \in \{0, 1\}$ and $x_1 \neq 0$.

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The word 101 is the binary expansion of $n = 5$:

$$5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1.$$

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The word $x_1 \dots x_k$ is *a hyperbinary expansion of n.*

The words 101 and 21 are both hyperbinary expansions of $n = 5$:

$$5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \qquad \textit{and} \qquad 5 = 2 \cdot 2^1 + 1.$$

The binary expansion of n is a particular hyperbinary expansion of n.

Applications in Number Theory

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Key point: $b(2n+1) = b(n)$ and $b(2n) = b(n) + b(n-1)$.

The Calkin-Wilf tree

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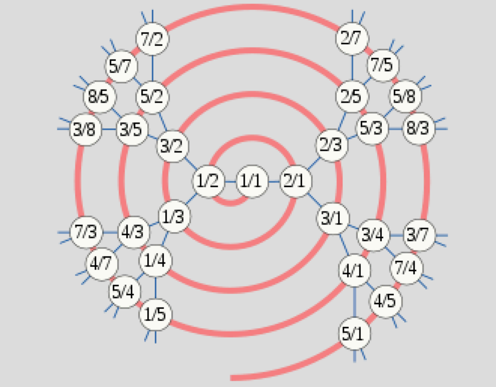

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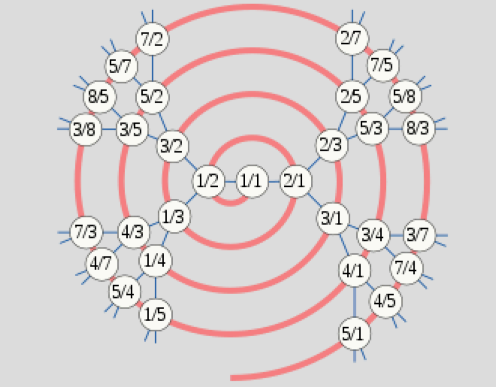

The Calkin-Wilf rooted tree

each a/b has two children: $a/(a+b)$ and $(a+b)/b$.

Applications to Number Theory

B. Bates and T. Mansour, *The q-Calkin-Wilf Tree*, J. Combin. Theory Ser. A **118** (2011), 1143–1151.

T. Mansour and M. Shattuck, *Two further generalizations of the Calkin–Wilf tree*, J. Comb. **2** (4) (2011), 507–524.

T. Mansour and M. Shattuck, *Generalized q-Calkin-Wilf trees and c-hyper m-expansions of integers*, J. Comb. Number Theory **7** (2015), no. 1, 1–12.

Combinatorics on words

Alessandro De Luca and C. Reutenauer, *Christoffel Words and the Calkin-Wilf Tree*, Electron. J. Combin. **18** (2) (2011-2), # P22.

J. Berstel and Aldo de Luca, *Sturmian words, Lyndon words and trees*, Theoret. Comput. Sci. **178** (1997), 171–203.

Alma D'Aniello, Aldo de Luca, Alessandro De Luca: *On Christoffel and standard words and their derivatives*, Theor. Comput. Sci. **658** (2017) 122–147.

The main problem

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'The' Problem

Let $n \neq 2^k - 1$. Given any hyperbinary expansion $\mathbf{x} \in \mathcal{H}(n)$. How 'far' is $\mathbf{x}$ from being binary?

Measuring non-binarity (1)

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In $\mathcal{H}(20)$ we have

$$\#_2(1212) > \#_2(2100) = \#_2(10012) > \#_2(10100) = 0.$$

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Note that a hyperbinary expansion has no 2's iff it is binary...

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In $\mathcal{H}(n)$, the (unique) hyperbinary expansion $\mathbf{n'}$ not containing 0's is minimal with respect to $<_{SL}$.

Some basic ingredients from the Theory of formal languages

- The *alphabet*: $\Sigma = \{0, 1, 2\}$.
- The *words*: the free monoid Σ^*.
- The equivalence relation $\sim$ that identifies two words in Σ^* differing only in zeros on the left-hand side.

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Re-writing systems

- The *rewrite rules*:

$$02 \rightarrow 10 \qquad \text{and} \qquad 12 \twoheadrightarrow 20.$$

- The *single-step reductions*:

$$\mathbf{x}\,0\,2\,\mathbf{y} \rightarrow \mathbf{x}\,1\,0\,\mathbf{y} \qquad \text{and} \qquad \mathbf{x}\,1\,2\,\mathbf{y} \twoheadrightarrow \mathbf{x}\,2\,0\,\mathbf{y} \qquad \forall\, \mathbf{x}, \mathbf{y} \in \Sigma^*.$$

- $\mathbf{u} \xrightarrow{*}_{\mathbf{R}} \mathbf{v}$. means that $\mathbf{u}$ and $\mathbf{v}$ in Σ^* are connected by a finite number $k > 0$ of single-step reductions.
- $\mathbf{u}$ is called an *ancestor of* $\mathbf{v}$, and $\mathbf{v}$ is *a descendant of* $\mathbf{u}$. If $k = 1$, we also say that $\mathbf{u}$ is a *parent of* $\mathbf{v}$, and $\mathbf{v}$ a *child of* $\mathbf{u}$.

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Crucial Proposition

Single-step reductions map elements in $\mathcal{H}(n)$ onto element in $\mathcal{H}(n)$.

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Proof.

For all $s \in \mathbb{N}$,
$$0 \cdot 2^s + 2 \cdot 2^{s-1} = 1 \cdot 2^s + 0 \cdot 2^{s-1};$$
$$1 \cdot 2^s + 2 \cdot 2^{s-1} = 2 \cdot 2^s + 0 \cdot 2^{s-1}.$$

□

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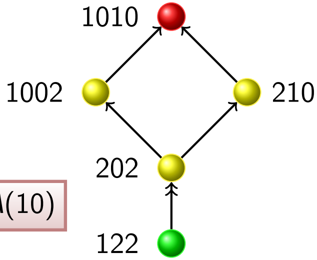

$A(10)$

02 $\rightarrow$ 10 and 12 $\twoheadrightarrow$ 20.

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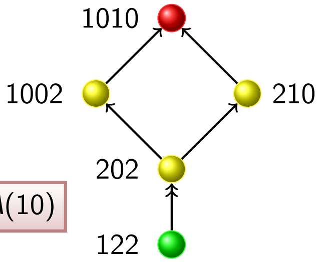

$\boxed{A(10)}$

- $|V(A(n))| = b(n)$.

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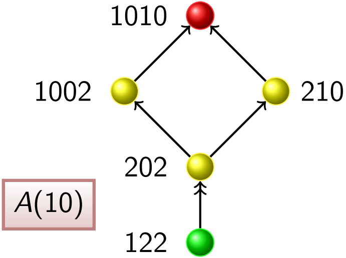

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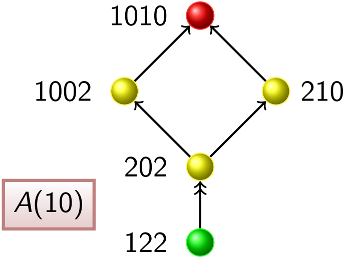

- $|V(A(n))| = b(n)$.
- The red ball is the unique binary expansion.
- The green ball is the unique expansion without 0's.

$02 \rightarrow 10$ and $12 \twoheadrightarrow 20$.

General properties of $A(n)$

- The graph $A(n)$ is simple: no node is a parent of itself and there is at most one arc connecting two nodes.
- The set of edges $\mathcal{E}(n)$ is empty if and only if $n = 2^k - 1$.
- $A(n)$ has a single root (i.e. a node with no ancestors) given by $\mathbf{n}'$, the unique hyperbinary expansion without 0's.
- No node in $A(n)$ is an ancestor of itself.
- $A(n)$ is a *flowchart*, i.e. is *pointed accessible*: for every node $\mathbf{n} \neq \mathbf{n}'$ there exists at least one path from the root to $\mathbf{n}$.
- The flowchart $A(n)$ has a *global sink*: all paths end to the binary expansion $\mathbf{n}''$, the unique node without children.

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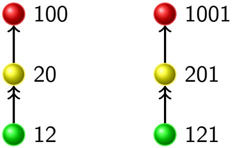
Retrieving the recursive functions

$$b(2n+1) = b(n)$$



$\boxed{A(4)}$ $\boxed{A(2 \cdot 4 + 1)}$

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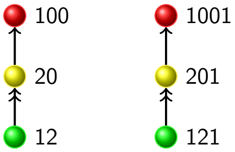
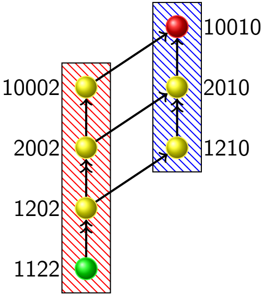


$\boxed{A(4)}$ $\boxed{A(2 \cdot 4 + 1)}$

$$b(2n) = b(n) + b(n-1)$$



$\boxed{A(18)}$

Generations

We say that $\mathbf{n} \in \mathcal{H}(n)$ belongs to the *k-th generation* if it is $k-1$ single-step reductions away from the common ancestor $\mathbf{n}'$ (the hyperbinary expansion without 0's).

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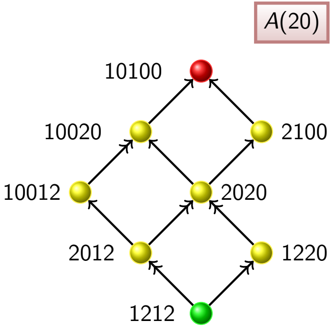

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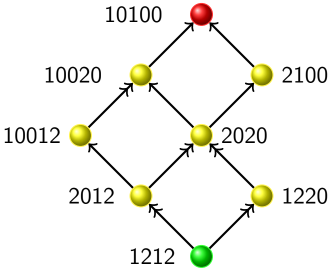
$A(20)$



1st gen.

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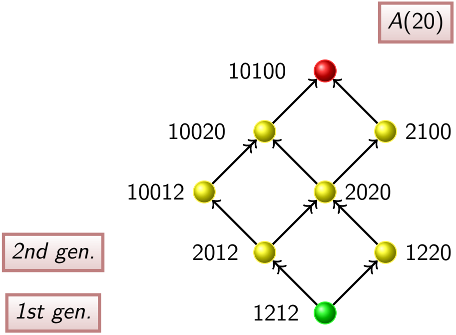

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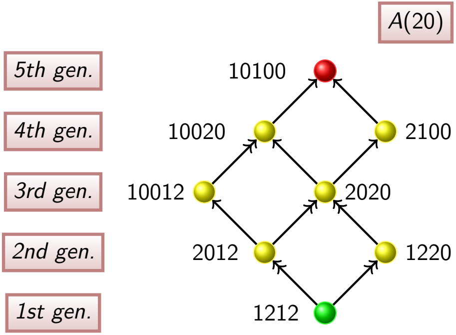

'Binarity' increases with 'youth level'.

The map ω

The youth level (i. e. the generation) can be equivalently computed through the map

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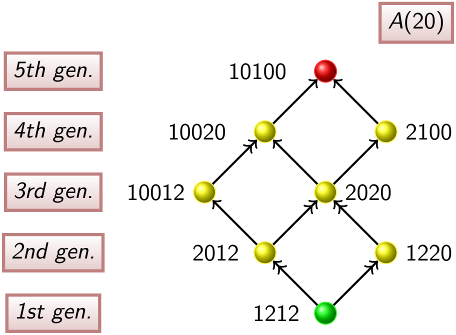

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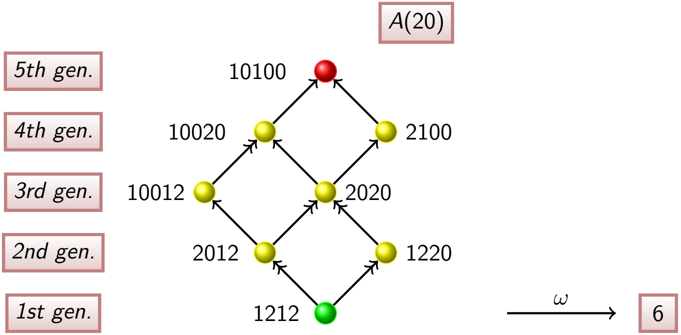

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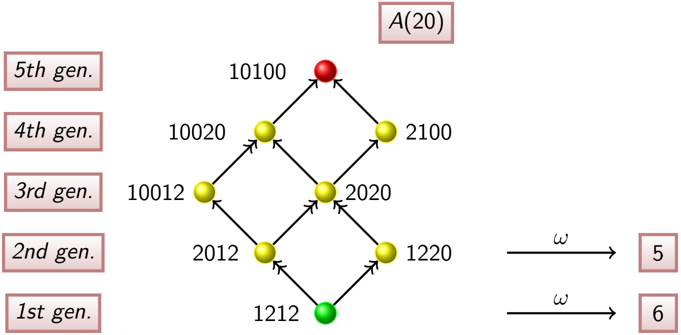

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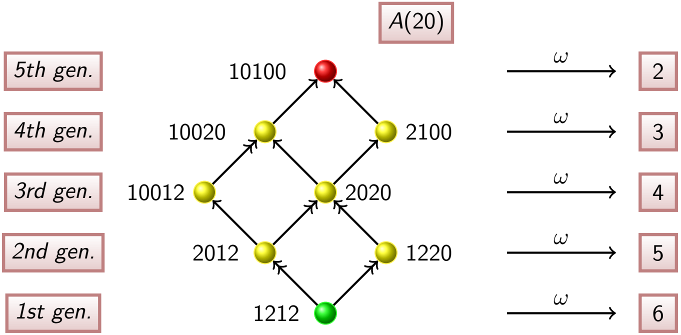

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Let $\mathbf{n}' \in \mathcal{H}(n)$ be the unique hyperbinary expansion without 0's, and let $\mathbf{n}'' \in \mathcal{H}(n)$ be the unique binary expansion.

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Theorem (D'A.-Brunetti (2019))

Let $\mathbf{n}' \in \mathcal{H}(n)$ be the unique hyperbinary expansion without 0's, and let $\mathbf{n}'' \in \mathcal{H}(n)$ be the unique binary expansion. Any hyperbinary expansion $\mathbf{n} \in \mathcal{H}(n)$ is $i(\mathbf{n})$ single-step transformations away from $\mathbf{n}''$ and $j(\mathbf{n})$ single-step transformations away from $\mathbf{n}'$, where

$$i(\mathbf{n}) = \omega(\mathbf{n}) - \omega(\mathbf{n}''), \quad \text{and} \quad j(\mathbf{n}) = \omega(\mathbf{n}') - \omega(\mathbf{n}).$$

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Theorem (D'A.-Brunetti (2019))

Let $\mathbf{n}' \in \mathcal{H}(n)$ be the unique hyperbinary expansion without 0's, and let $\mathbf{n}'' \in \mathcal{H}(n)$ be the unique binary expansion.

$$b(n) \geq \omega(\mathbf{n}'') - \omega(\mathbf{n}') + 1.$$

The equality holds if and only if the graph $A(n)$ is a tree.

Few theorems

Theorem (D'A.-Brunetti (2019))

The graph $A(n)$ is a tree if and only if there exists $(s, t) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that

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- $\min\{n \mid A(n) \text{ is not a tree}\} = 10.$
- the difference $b(n) - \omega(\mathbf{n''}) + \omega(\mathbf{n'}) - 1$ increases with the *cyclomatic number* $\nu(n)$.

The cyclomatic number

Definition

Let $V(G)$ and $E(G)$ be the set of vertices and edges respectively of a connected graph G. The cyclomatic number $\nu(G)$ of G is given by

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Theorem (D'A.-Brunetti (2019))

Let n be a positive integer not equal to $2^k - 1$. The following formula holds.

$$\nu(A(n)) = \sum_{\mathbf{n} \neq \mathbf{n}''} (o(\mathbf{n}) - 1),$$

where $o(\mathbf{n})$ is the outdegree of $\mathbf{n}$, i.e. the number of its children.

Unicyclic graphs and open problems

Theorem (D'A.-Brunetti (2019))

$$\nu(A(n)) = 1 \iff n = 2^{\ell}(12 \pm 1) - 1 \quad \text{for a suitable } \ell \geq 0.$$

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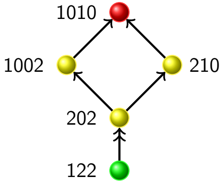
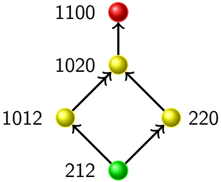
$$\nu(A(10)) = \nu(A(12)) = 1, \quad b(10) = b(12) = 5 \quad \text{but} \quad A(10) \ncong A(12).$$

(look at the edges!)



$A(10)$



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Unicyclic graphs and open problems

Conjecture

Suppose $m > n \geq 1$.
$A(n) \cong_{dec} A(m)$ if and only if $m = 2^{\ell}(n+1) - 1$ for a suitable $\ell > 0$.

Unicyclic graphs and open problems

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Suppose $m > n \geq 1$.
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$$
\begin{array}{llll}
A(1) & \cong_{dec} A(3) & \cong_{dec} A(7) & \cong_{dec} \cdots \\
A(2) & \cong_{dec} A(5) & \cong_{dec} A(11) & \cong_{dec} \cdots \\
A(4) & \cong_{dec} A(9) & \cong_{dec} A(19) & \cong_{dec} \cdots \\
 & \dots & \dots & \dots \\
A(2q) & \cong_{dec} A(4q+1) & \cong_{dec} A(8q+3) & \cong_{dec} \cdots
\end{array}
$$

SandGAL

Cremona, June 10-13, 2019

Thank you!