Groups generated by bireversible Mealy automata: a combinatorial explosion

SANDGAL 2019

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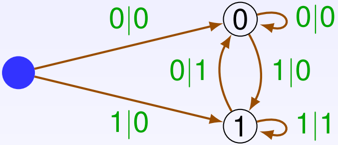
Action on infinite words

- $\Sigma = \{0, 1, 2\}$ $\qquad 0 \leftrightarrow 1$
  $012112002120121021\ldots \mapsto 102002112021020120\ldots$

Action on infinite words

- $\Sigma = \{0, 1, 2\}$ $\quad 0 \leftrightarrow 1$
  $0121120021201210 21\ldots \mapsto 1020021120210201 20\ldots$

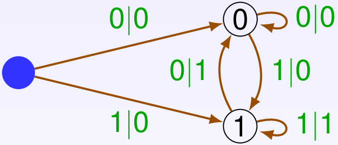
- $\Sigma = \{0, 1\}$ $\quad \mathbf{w} \mapsto 0\mathbf{w}$



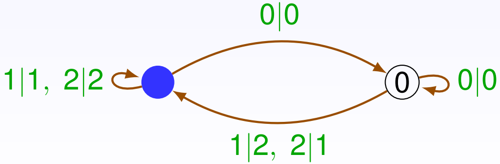
Action on infinite words

- $\Sigma = \{0, 1, 2\} \qquad 0 \leftrightarrow 1$
  $012112002120121021\ldots \mapsto 102002112021020120\ldots$

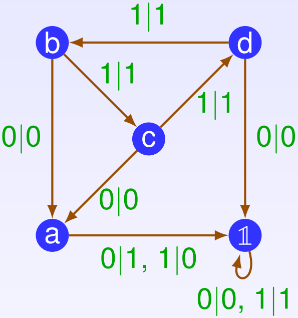
- $\Sigma = \{0, 1\} \qquad \mathbf{w} \mapsto 0\mathbf{w}$



- $\Sigma = \{0, 1, 2\} \qquad 02 \mapsto 01,\ 01 \mapsto 02$



Groups generated by Mealy automata



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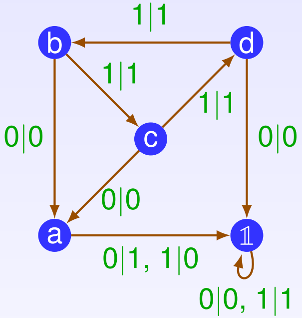
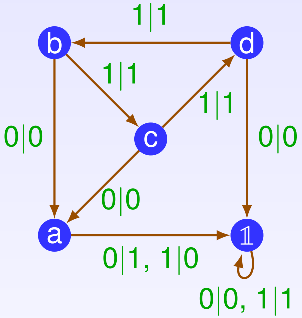


$\rho_d : \quad 1000\mathbf{w} \mapsto 1010\mathbf{w}$

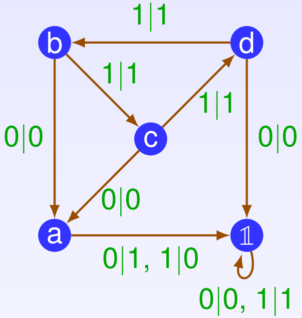
Groups generated by Mealy automata



$$\rho_d : \quad 1000\mathbf{w} \mapsto 1010\mathbf{w}$$

$$\Sigma^\omega \to \Sigma^\omega$$

Groups generated by Mealy automata



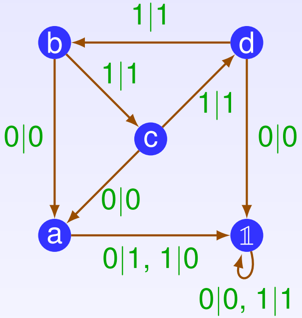
ρ_d : 1000**w** $\mapsto$ 1010**w**

$\Sigma^\omega \to \Sigma^\omega$

- deterministic [functional]
- complete [defined all over Σ^ω]

$\langle \mathcal{A} \rangle_+$ semigroup

Groups generated by Mealy automata



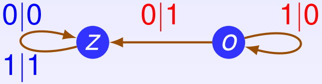
ρ_d : $1000\mathbf{w} \mapsto 1010\mathbf{w}$

$\Sigma^\omega \to \Sigma^\omega$

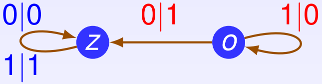
- deterministic [functional]
- complete [defined all over Σ^ω]
- the states permute the alphabet [for groups]

$\langle \mathcal{A} \rangle_+$ semigroup
$\langle \mathcal{A} \rangle$ group

An example



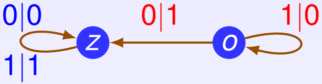
An example



ρ_z : $111001010100\ldots \mapsto 111001010100\ldots$

ρ_o : $111001010100\ldots \mapsto 000101010100\ldots$

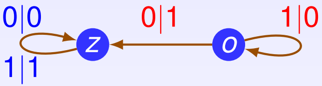
An example



$$\rho_z : 111001010100\ldots \mapsto 111001010100\ldots$$

$$\rho_o : 111001010100\ldots \mapsto 000101010100\ldots \qquad + 1$$

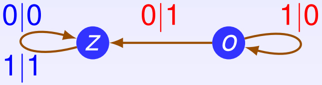
An example



ρ_z : 111001010100 . . . $\mapsto$ 111001010100 . . . $+ 0$

ρ_o : 111001010100 . . . $\mapsto$ 000101010100 . . . $+ 1$

An example



$\mathbb{Z}$

$$\rho_z : 111001010100\ldots \mapsto 111001010100\ldots \qquad +0$$

$$\rho_o : 111001010100\ldots \mapsto 000101010100\ldots \qquad +1$$

How to go to infinity? 1

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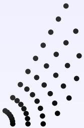

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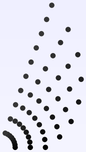

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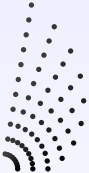

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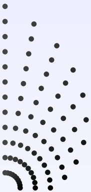

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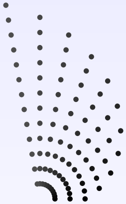

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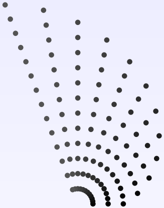

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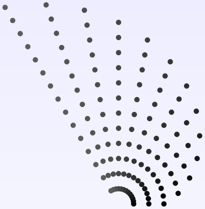

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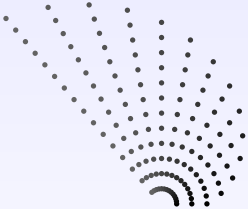

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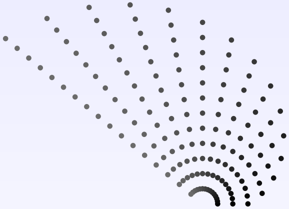

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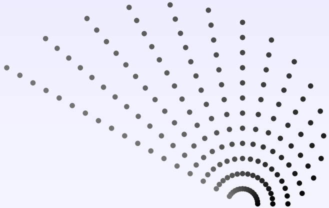

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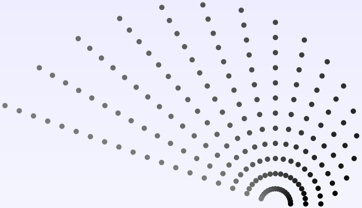

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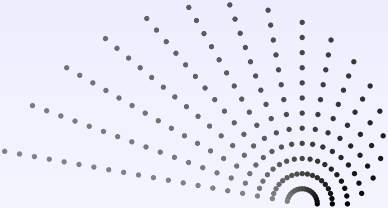

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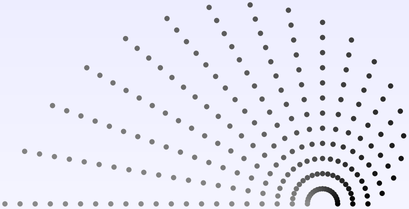

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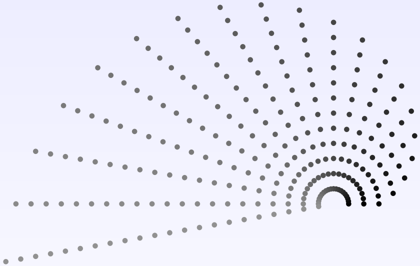

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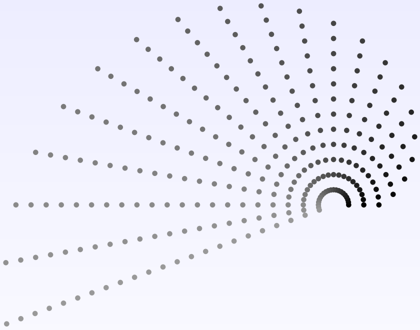

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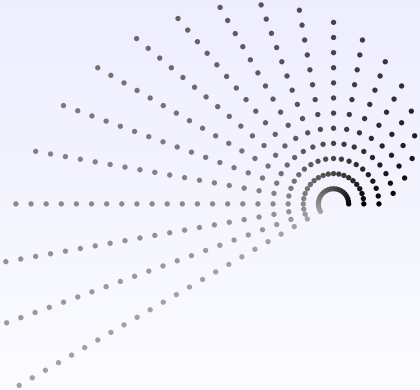

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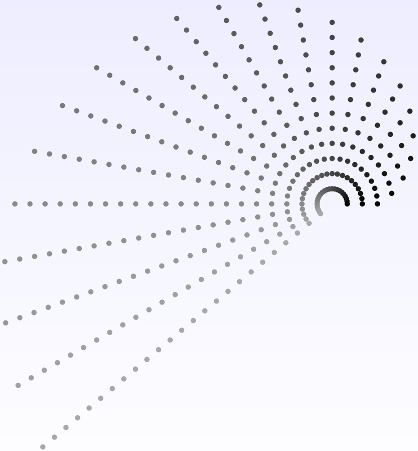

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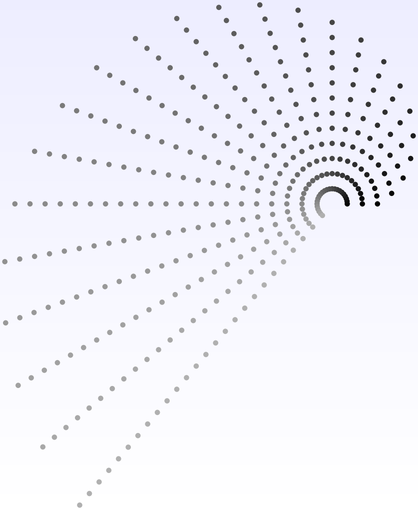

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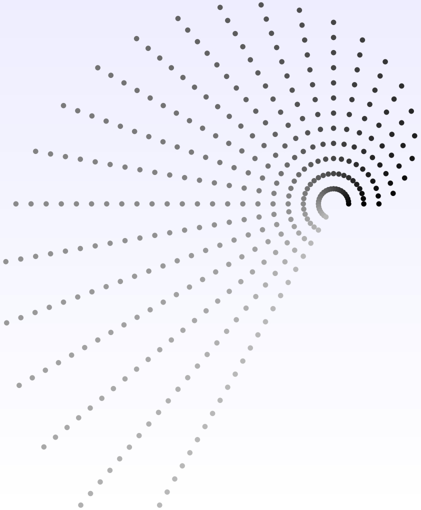

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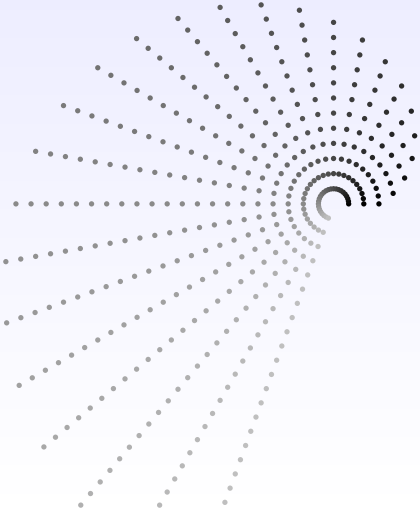

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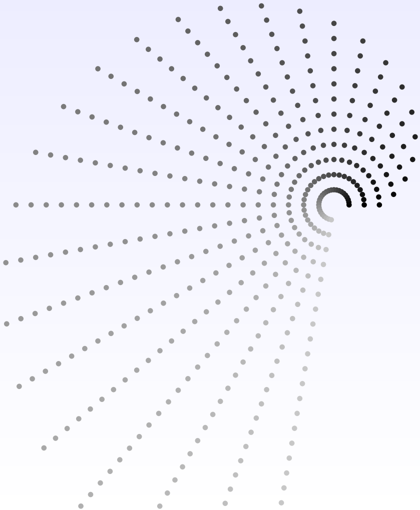

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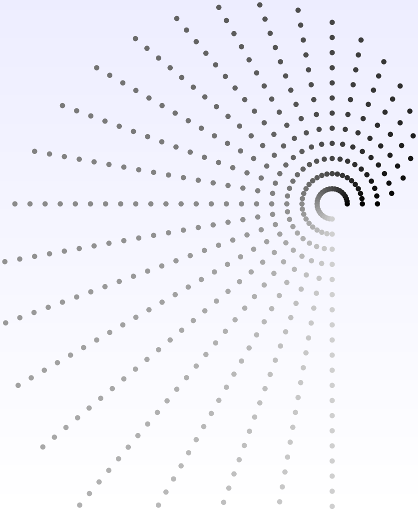

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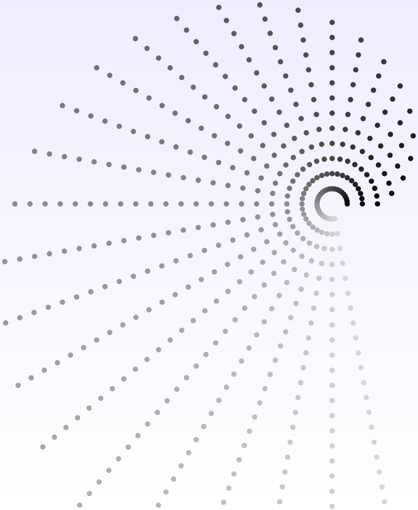

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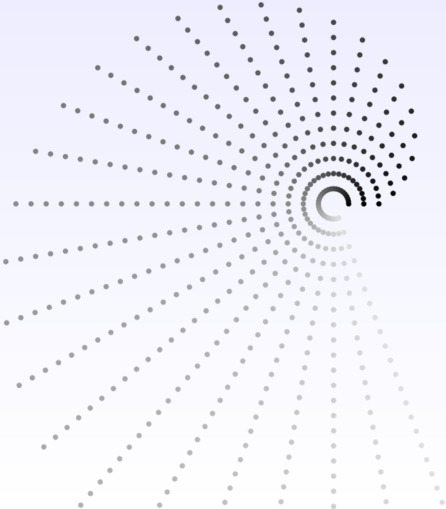

How to go to infinity? 1

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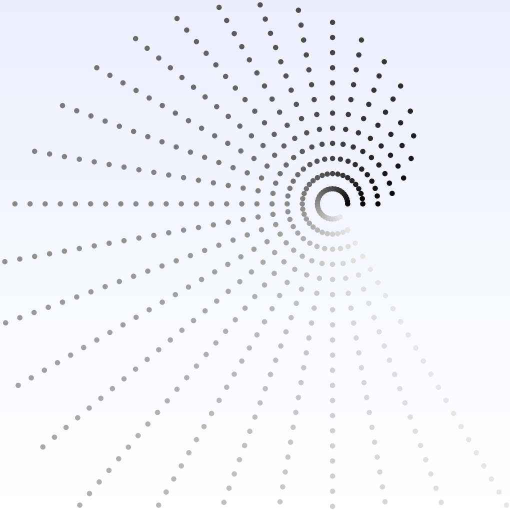

How to go to infinity? 1

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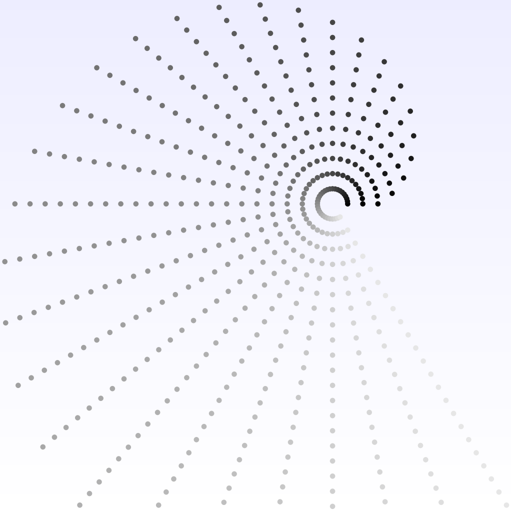

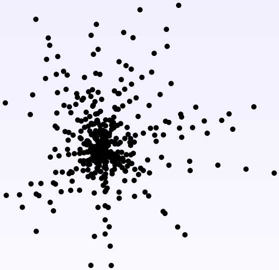

How to go to infinity? 1

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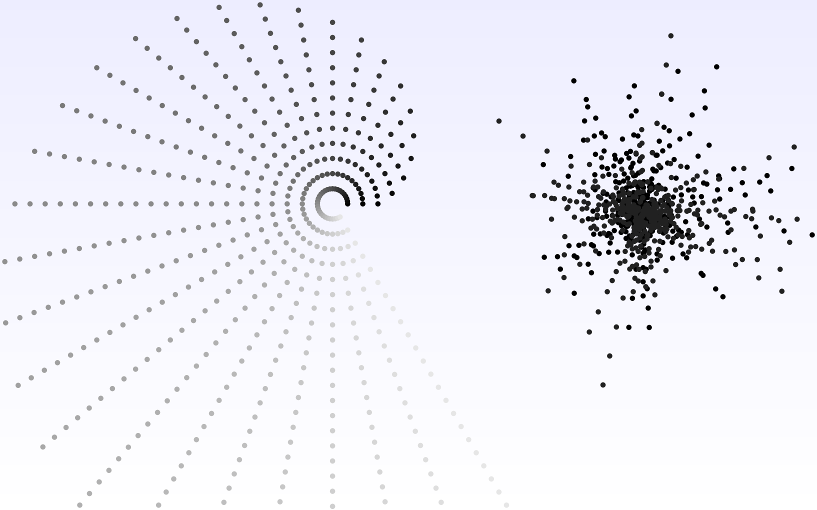

How to go to infinity? 1

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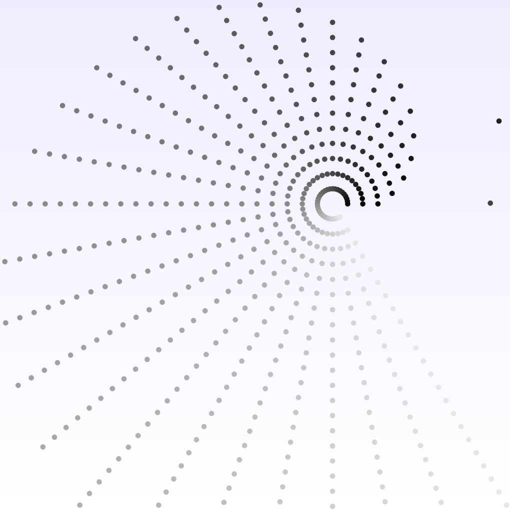

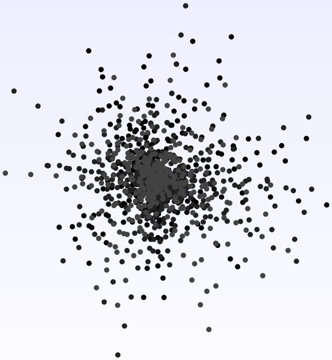

How to go to infinity? 1

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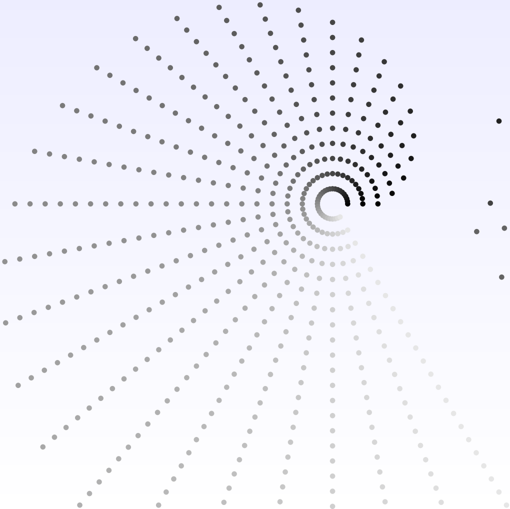

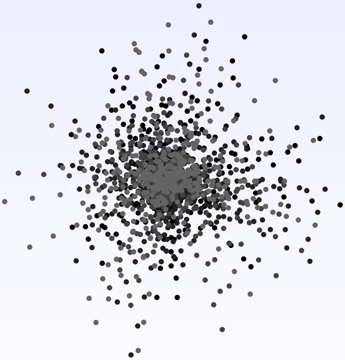

How to go to infinity? 1

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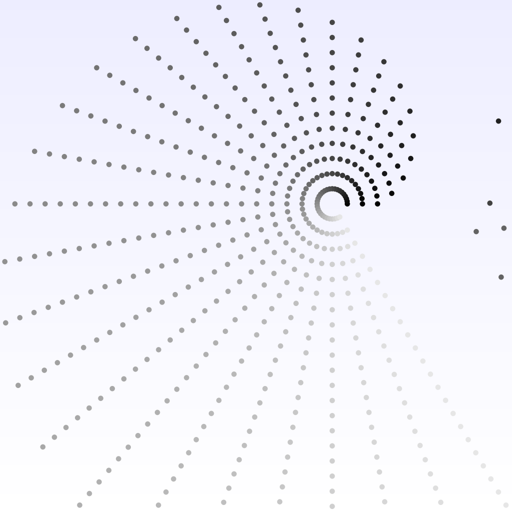

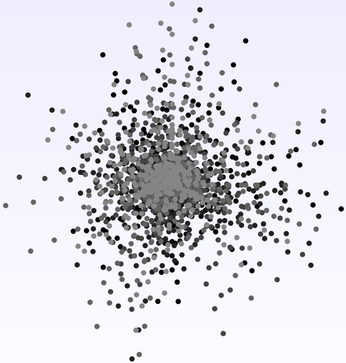

How to go to infinity? 1

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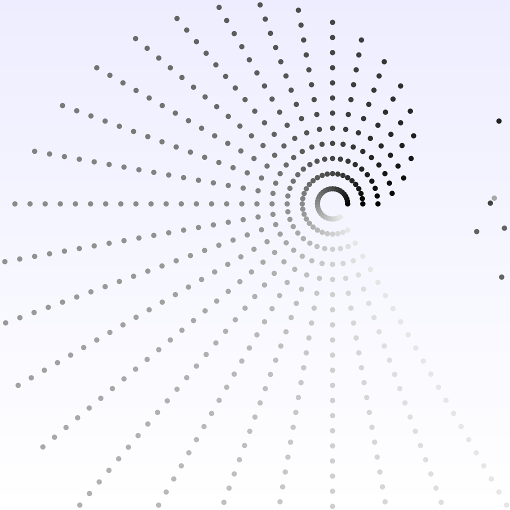

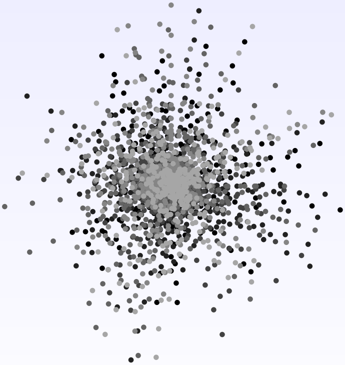

How to go to infinity? (1)

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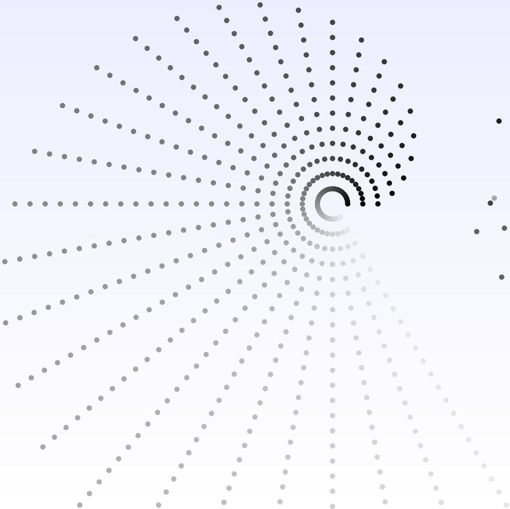

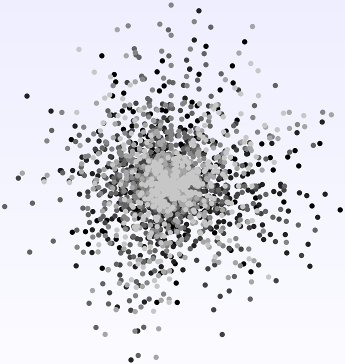

How to go to infinity? 1

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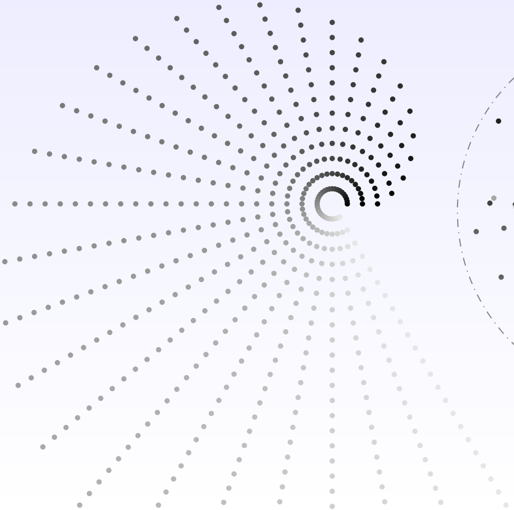

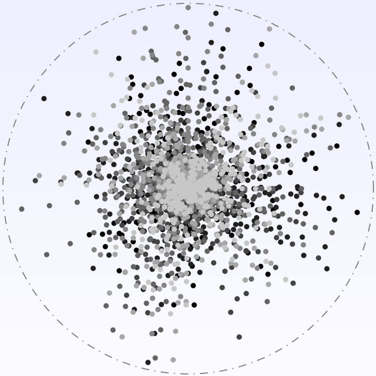

How to go to infinity? 1

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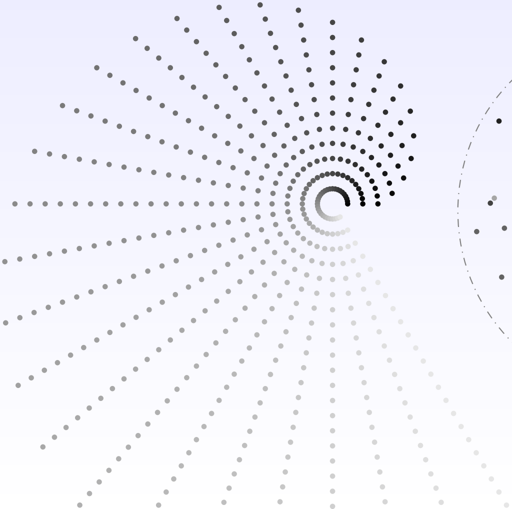
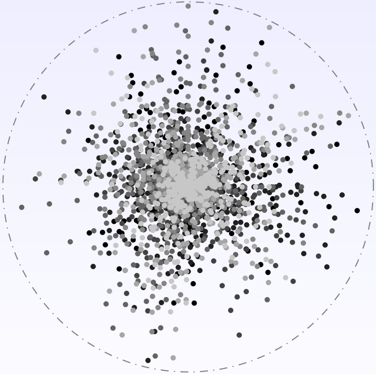
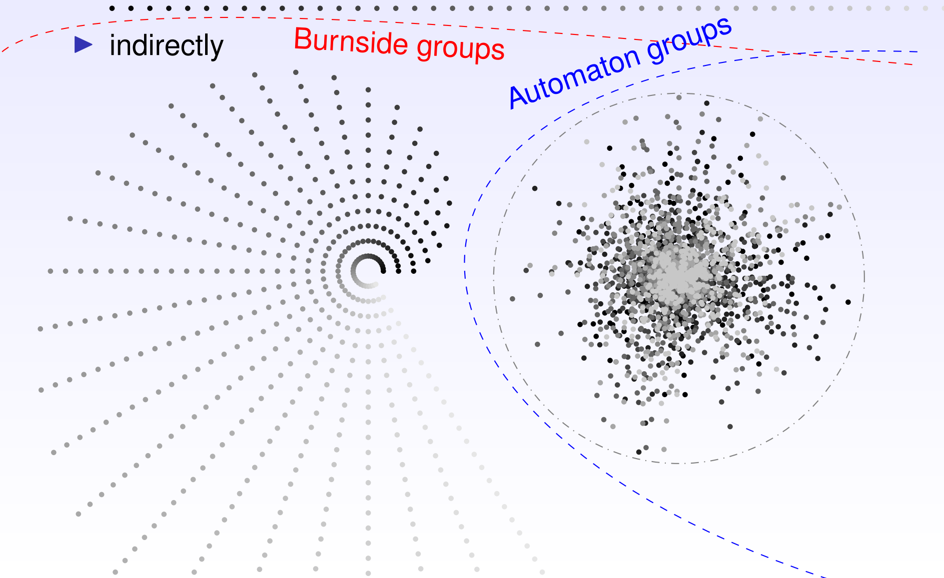
Burnside groups





How to go to infinity? ①

- directly
- indirectly



How to go to infinity? 2

- slow
- fast

How to go to infinity? 2

- slow
- .
- fast

How to go to infinity? 2

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- fast

How to go to infinity? 2

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- fast

How to go to infinity? 2

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How to go to infinity? 2

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- fast

How to go to infinity? 2

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How to go to infinity? 2

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- fast

How to go to infinity? ②

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- fast

How to go to infinity? 2

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- fast

How to go to infinity? ②

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How to go to infinity? 2

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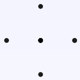

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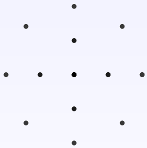
- fast



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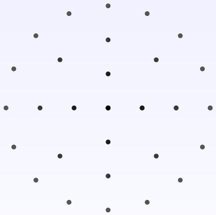

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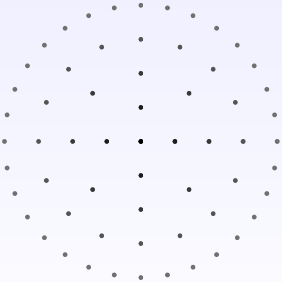

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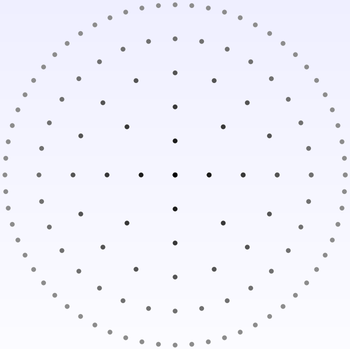

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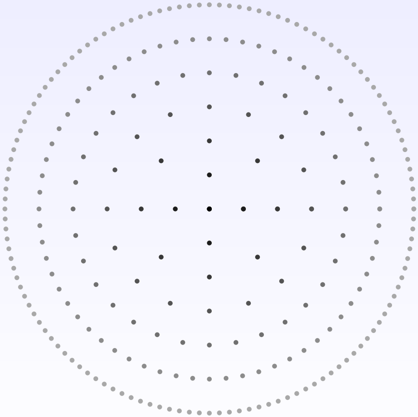

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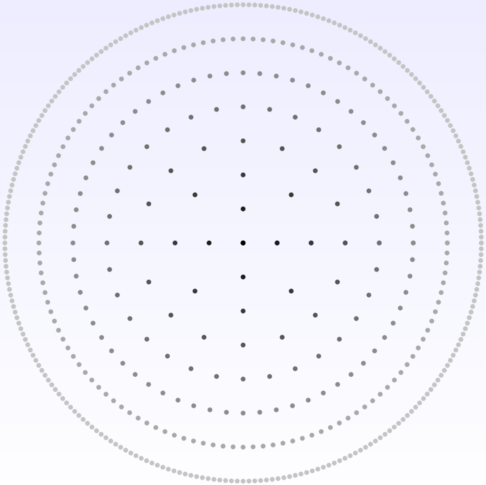
- slow

- fast



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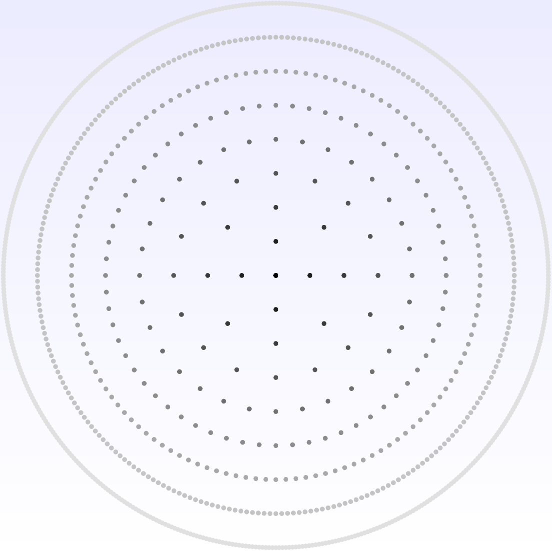
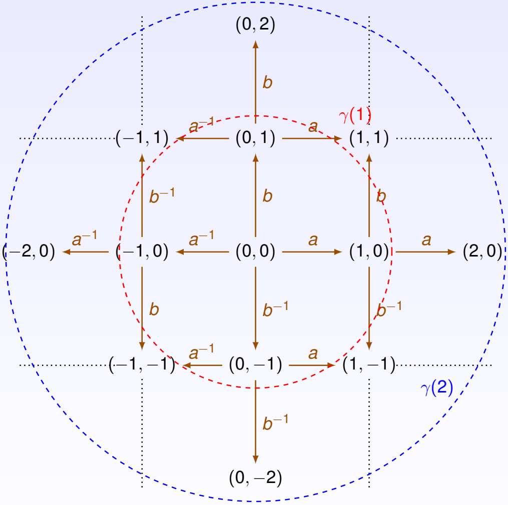


Growth

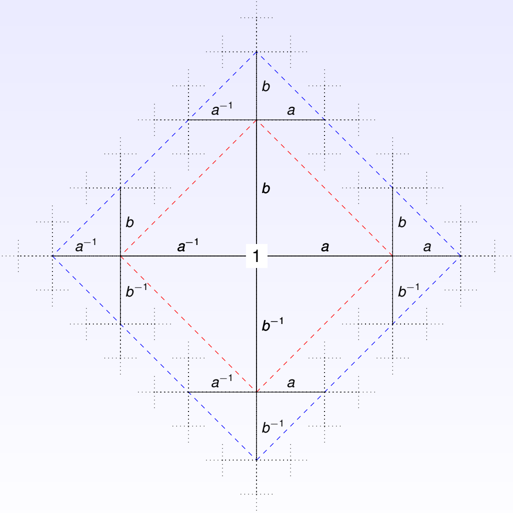
$$\mathbb{Z}^2 = \langle a = (0,1), b = (1,0) \mid ab = ba \rangle$$



$$\gamma(0) = 1$$
$$\gamma(1) = 5$$
$$\gamma(2) = 13$$
$$\vdots$$
$$\gamma(n) = 2n^2 + 2n + 1$$

Growth

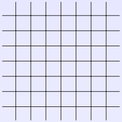
$$\mathbb{F}_2 = \langle a, b \rangle$$



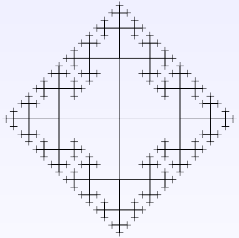
$$\gamma(0) = 1$$
$$\gamma(1) = 5$$
$$\gamma(2) = 17$$
$$\vdots$$
$$\gamma(n) = 2 \times 3^n - 1$$

bounded growth: finite groups

polynomial growth: $\mathbb{Z}^d$, abelian groups, ...



exponential growth: $\mathbb{F}_d$, ...



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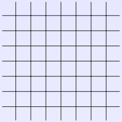
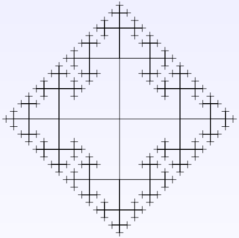


Milnor

1968

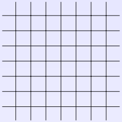
And in between?

exponential growth: $\mathbb{F}_d$, ...



bounded growth: finite groups

polynomial growth: $\mathbb{Z}^d$, abelian groups, ...



Milnor

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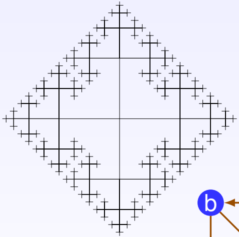
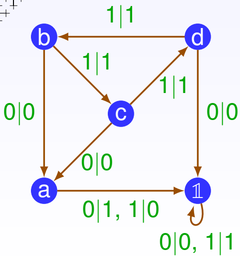


exponential growth: $\mathbb{F}_d$, ...



Grigorchuk

1983

It's possible!

Some results on groups of intermediate growth

Grigorchuk'83

There exists a group of intermediate growth.

Bartholdi, Erschler'2014

For many "good" $\alpha \leq 1$, there exists a group of growth $\sim \exp(n^\alpha)$.

Nekrashevich'2018

A first example of group of intermediate growth which does not involve an automaton group.

Some questions on groups of intermediate growth

Open question

Is it decidable if an automaton group has intermediate growth?

K., ICALP'18

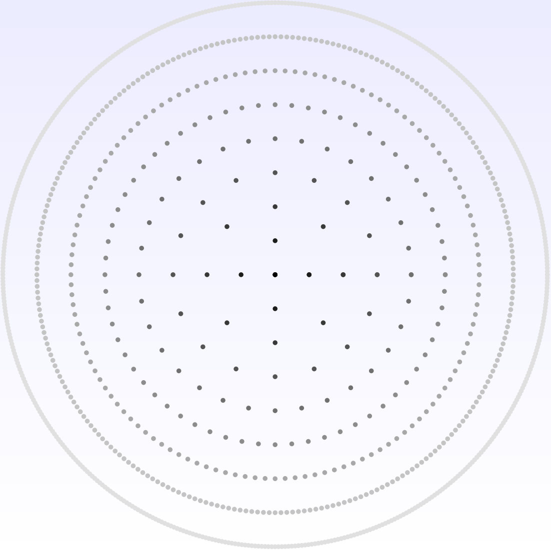
$\mathcal{A}$ bireversible:

$$\exists g \in \langle \mathcal{A} \rangle \text{ of infinite order } \implies \langle \mathcal{A} \rangle \text{ has exponential growth}$$

How to go to infinity?

- directly (1)

- fast (2)



Bireversible automata

- states permute letters [group]
- input letters permute states
- output letters permute states

Bireversible automata

- states permute letters [group]
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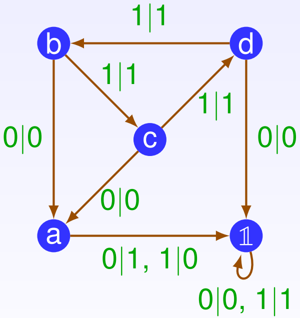

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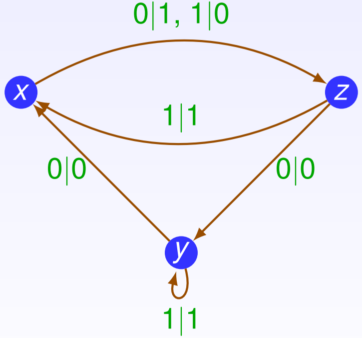


Why looking at bireversible automata?

Property

All connected component are strongly connected.

Lemma

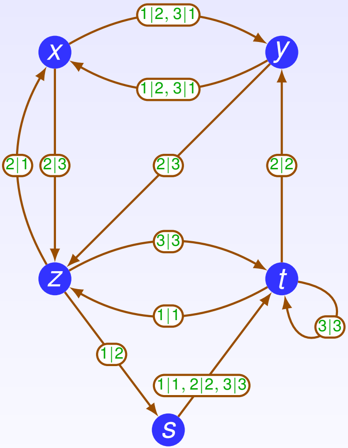
If $\mathcal{A}$ connected, its Nerode classes have all the same cardinality.

Proposition

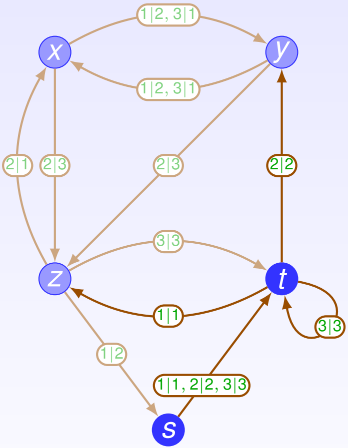
If $\mathcal{A}$ and $\mathcal{B}$ connected and minimal on the same alphabet

$$\#\mathcal{A} \neq \#\mathcal{B} \implies \forall p \in \mathcal{A}, \forall q \in \mathcal{B}, \rho_p \neq \rho_q$$

Nerode classes and minimization



Nerode classes and minimization



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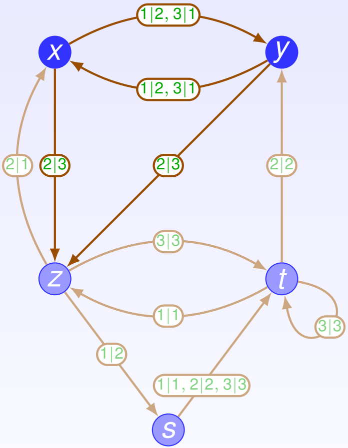

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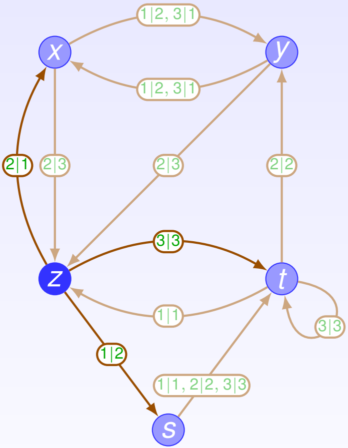

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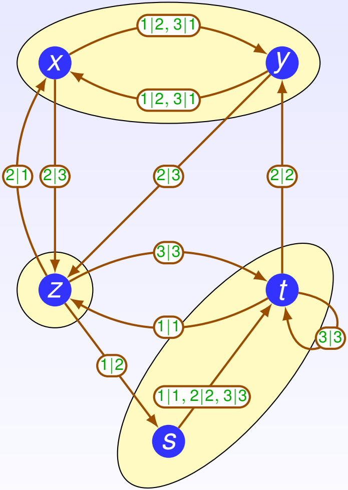
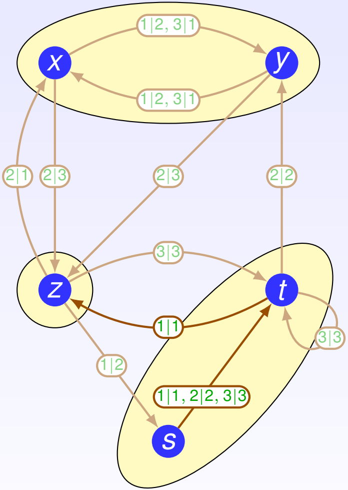


$$\rho_{x_{|\Sigma}} = \rho_{y_{|\Sigma}}$$

$$\rho_{s_{|\Sigma}} = \rho_{t_{|\Sigma}}$$

Nerode classes and minimization



$$\rho_{x_{|\Sigma}} = \rho_{y_{|\Sigma}}$$

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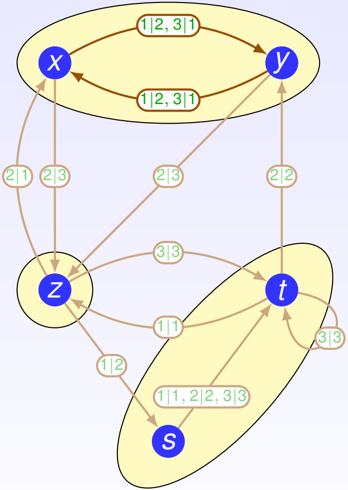
Nerode classes and minimization



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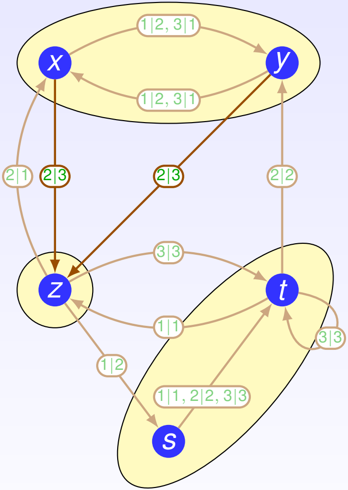

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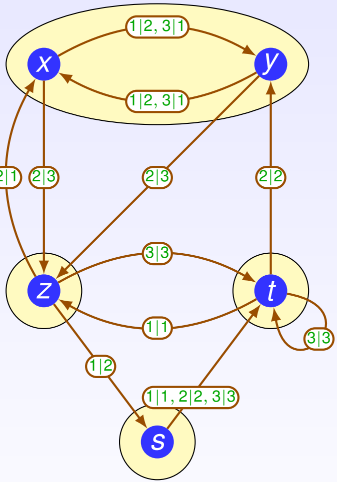
Nerode classes and minimization



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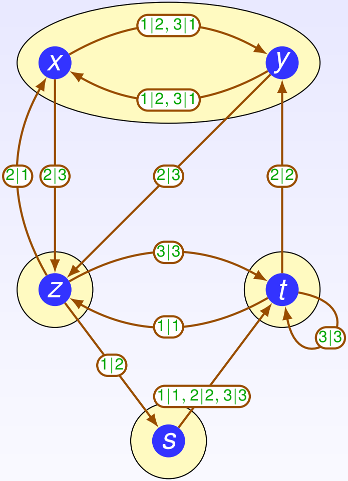
Nerode classes and minimization



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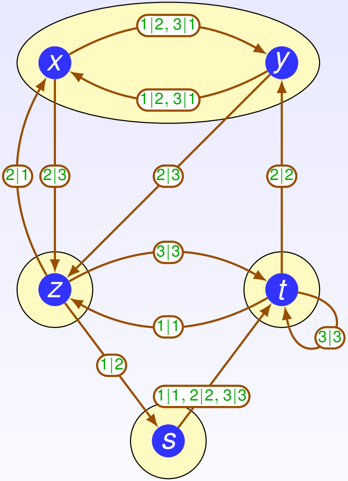
$\rightsquigarrow$ classical minimisation:

$$[\![x]\!] = [\![y]\!] \iff \rho_{x_{|\Sigma^*}} = \rho_{y_{|\Sigma^*}}$$

Nerode classes and minimization



$\rightsquigarrow$ classical minimisation:

$$[\![x]\!] = [\![y]\!] \iff \rho_{x_{|\Sigma^*}} = \rho_{y_{|\Sigma^*}}$$

- $\mathfrak{m}(\mathcal{A}) \simeq \mathfrak{m}(\mathcal{B}) \implies \langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$
- $\mathfrak{m}(\mathcal{A}^k) \simeq \mathfrak{m}(\mathcal{A}^\ell)$ with $k \neq \ell \implies \langle \mathcal{A} \rangle$ finite

Why looking at bireversible automata?

Property

All connected component are strongly connected.

Lemma

If $\mathcal{A}$ connected, its Nerode classes have all the same cardinality.

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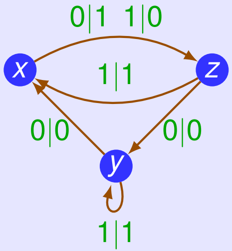
Why looking at bireversible automata?

Proposition

ρ_q fin. ord. $\Leftrightarrow (\#\,\mathrm{cc}(q^k))_k$ bounded $\Leftrightarrow (\#\mathfrak{m}(\mathrm{cc}(q^k)))_k$ bounded

Construction

$\mathcal{A}$



$$\rho_{x_i} : \Sigma^* \to \Sigma^*$$
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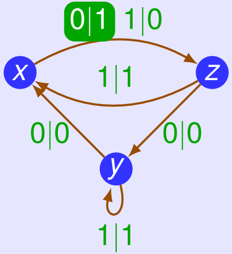

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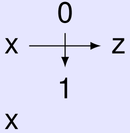

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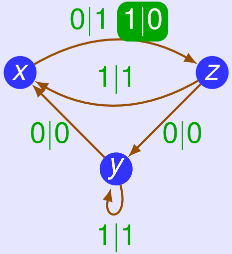

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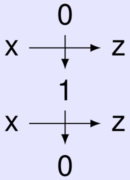

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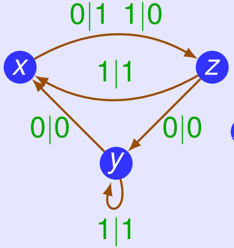

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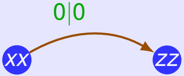

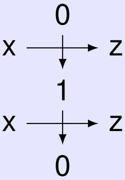

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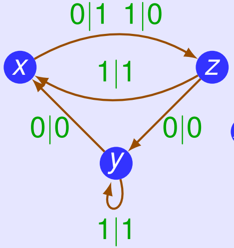
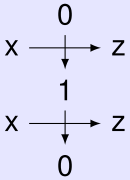
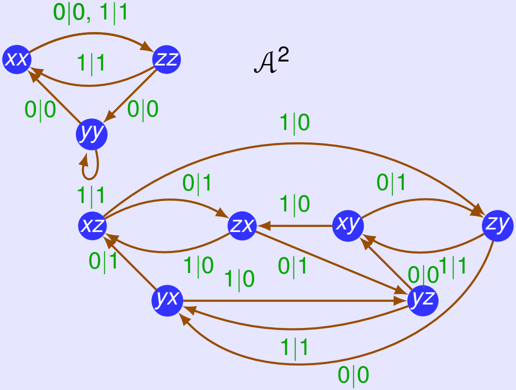
$\mathcal{A}$





$\mathcal{A}^2$



Some questions on groups of intermediate growth

Open question

Is it decidable if an automaton group has intermediate growth?

K., ICALP'18

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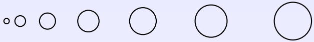

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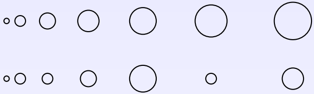

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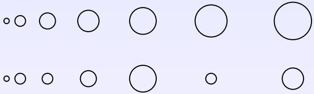

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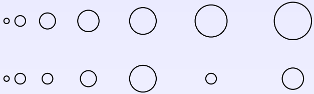

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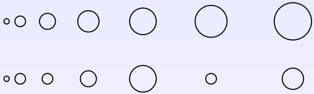

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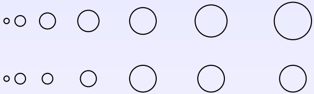

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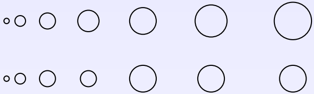

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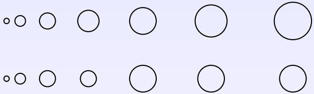

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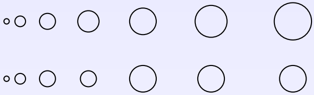

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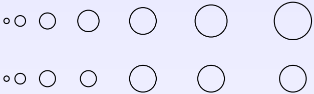

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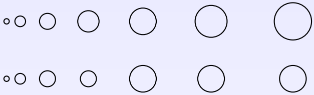

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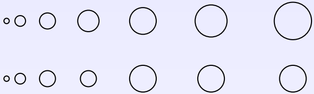

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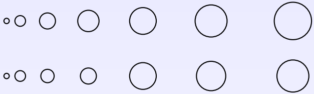

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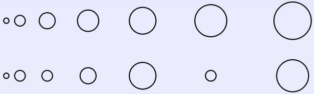

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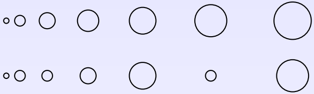

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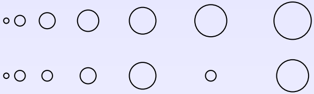

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$$[\![\, q^k \bullet]\!] = [\![q^k]\!]\cap\mathrm{cc}(q^k)\cap\Sigma^* q$$

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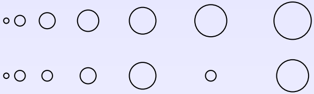

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$\llbracket q^k\rrbracket q \subseteq \llbracket q^{k+1}\rrbracket$	$(\llbracket q^k\rrbracket\cap\mathrm{cc}(q^k))q\ ?\ \llbracket q^{k+1}\rrbracket\cap\mathrm{cc}(q^{k+1})$

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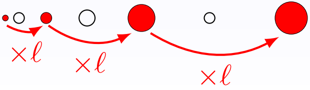

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Corollary

No infinite virtually nilpotent group can be generated by a bireversible Mealy automaton.

Perspective

Bireversible automata

Is it possible to generate a group of polynomial or intermediate growth with a bireversible automaton?

Automaton groups

Is the growth of an automaton group decidable?