Boundedly finite conjugacy classes of tensors

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"Semigroups and Groups, Automata, Logics"

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This is a joint work[1] with **Raimundo Bastos**.

R. Bastos, C. Monetta,
Boundedly finite conjugacy classes of tensors, submitted.

[1] Funded by GNSAGA

Non-abelian Tensor Square of a Group

Let G be a group. We denote by $G \otimes G$ the group generated by the symbols $g \otimes h$, with $g, h \in G$, which satisfy the following conditions:

- $gh \otimes x = (g^h \otimes x^h)(h \otimes x)$
- $g \otimes hx = (g \otimes x)(g^x \otimes h^x)$

for every $g, h, x \in G$.

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Topological importance

Brown and **Loday** presented a topological significance for the non-abelian tensor square.

R. Brown, J.-L. Loday,
Van Kampen theorems for diagrams of spaces, Topology, **26** (1987), 311-335.

They showed that the third homotopy group of the suspension of an Eilenberg-MacLane space K(G,1) satisfies

$$\pi_3 SK(G,1) \simeq \mu(G)$$

where $\mu(G)$ is the kernel of the derived map

$$k : G \otimes G \to G',$$

$g \otimes h \mapsto [g,h] = g^{-1}h^{-1}gh.$

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A related construction of Rocco

In this presentation we will deal with some algebraic aspects related to the non-abelian tensor square.

More precisely, we will focus on the similarities between commutators and tensors.

Given a group G, let G^φ be an isomorphic copy of G ($g \mapsto g^\varphi$).

$$\nu(G) := \langle G, G^\varphi \mid [g_1, {g_2}^\varphi]^{g_3} = [{g_1}^{g_3}, ({g_2}^{g_3})^\varphi] = [g_1, {g_2}^\varphi]^{{g_3}^\varphi}, \quad g_i \in G \rangle.$$

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Non-abelian tensor square as a subgroup

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is the commutator connection:

Proposition (Rocco, 1991)

The map $\Phi : G \otimes G \to [G, G^{\varphi}]$, defined by $g \otimes h \mapsto [g, h^{\varphi}]$, for every $g, h \in G$, is an isomorphism.

Since $[G, G^{\varphi}] \leqslant \nu(G)'$, to investigate properties of $G \otimes G$, one can look at the commutators in the group $\nu(G)$.

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They compute the non-abelian tensor square of all non-abelian groups of order up to 30 using Tietze transformations.

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Donadze, **Ladra** and **Thomas**, proved that if a group G belongs to some class of groups, then so $G \otimes G$ does.

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Remark

Let $\mathfrak{X}$ be a class of groups "closed" under taking **subgroups**, **direct products** *and* **central extensions** *of its elements.*

If $G \in \mathfrak{X}$, then $\nu(G) \in \mathfrak{X}$.

One can consider the homomorphism $\rho : \nu(G) \to G$ defined by $g \mapsto g$ and $g^{\varphi} \mapsto g$. The kernel of this map is denoted by $\Theta(G)$. Then we have

$$\frac{\nu(G)}{\Theta(G)} \simeq G \qquad \text{and} \qquad \frac{\nu(G)}{[G, G^{\varphi}]} \simeq G \times G$$

and $\mu(G) := [G, G^{\varphi}] \cap \Theta(G) \leq Z(\nu(G))$, is a central subgroup.

Then $\nu(G)/\mu(G)$ is isomorphic to a subgroup of $G \times G \times G$. It follows that $\nu(G) \in \mathfrak{X}$.

Examples of classes: nilpotent, supersoluble, soluble, locally nilpotent, locally soluble, Engel, ...

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Elements with finitely many conjugates

Given a group G, we use the following notation:

$$\Gamma(G) := \{[g,h] \mid g,h \in G\} \quad \textbf{and} \quad x^G := \{x^g \mid g \in G\}$$

A group G is said to be a BFC-group if there exists a positive integer n such that $|x^G| \leq n$ for every $x \in G$.

If n is the least upper bound of the size of the conjugacy classes, then we say that G is a n-BFC-group.

Remark

If $|\Gamma(G)| = n$ is finite, then G is a k-BFC-group for some $k \leq n$.

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Theorem (Neumnn, 1951)

A group G is a BFC-group $\iff$ G' is finite $\iff$ $\Gamma(G)$ is finite

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Wiegold obtained a quantitative version of the previous theorem.

Theorem (Wiegold, 1958)

If G is a BFC-group with least upper bound of the conjugacy classes given by the positive integer n, then $|G'| \leq n^{\frac{1}{2}n^4(\log_2 n)^3}$.

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The best bound known is due to Guralnick and Maróti.

Theorem (Guralnick, Maróti, 2011)

Let G be an n-BFC-group with $n > 1$. Then $|G'| < n^{\frac{1}{2}(7+\log_2 n)}$.

R. M. Guralnick, A. Maróti,
Advances in Mathematics 226 (2011) 298-308, Advances in Mathematics, **226** (2011), 298-308.

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Let G be an n-BFC-group with $n > 1$. Then $|G'| < n^{\frac{1}{2}(7+\log_2 n)}$.

R. M. Guralnick, A. Maróti,
Advances in Mathematics 226 (2011) 298-308, Advances in Mathematics, **226** (2011), 298-308.

Tensor version

We denote by $T_{\otimes}(G)$ the set of all tensors, i.e.,

$$T_{\otimes}(G) = \{[g, h^{\varphi}] \mid g, h \in G\}.$$

Theorem (Bastos, Nakaoka, Rocco, 2018)

The non-abelian tensor square $[G, G^{\varphi}]$ is finite if and only if $T_{\otimes}(G)$ is finite.

R. Bastos, I. N. Nakaoka, N. R. Rocco, *Finiteness conditions for the non-abelian tensor product of groups*, Monatsh. Math., **187** (2018), 603-615.

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Commutators with finitely many conjugates

Recently, **Dierings** and **Shumyatsky** coped with a similar problem for commutators.

Theorem (Dierings, Shumyatsky, 2018)

Let n be a positive integer and assume that $|x^G| \leq n$ *for every commutator* $x \in \Gamma(G)$*.*
Then the second derived subgroup G'' *is finite with n-bounded order.*

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What about tensors?

Question

Le n be a positive integer. Assume that $\left|\alpha^{\nu(G)}\right| \leq n$ for any $\alpha \in T_{\otimes}(G)$. Is then $([G, G^{\varphi}])'$ finite?

Since $[G, G^{\varphi}] \leq \nu(G)'$, then $([G, G^{\varphi}])' \leq \nu(G)''$.

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Theorem (Bastos, M., 2019)

Let n be a positive integer. Suppose that $\left|\alpha^{\nu(G)}\right| \leq n$ *for every* $\alpha \in T_{\otimes}(G)$*. Then* $\nu(G)''$ *is finite with n-bounded order.*

If $[G, G^{\varphi}]$ is finite $\not\Rightarrow$ G is a finite group.

For instance, the Prüfer group C_{p^∞} is an example of an infinite group such that the non-abelian tensor square $[C_{p^\infty}, (C_{p^\infty})^{\varphi}]$ is trivial (and so, finite).

However, **Parvizi** and **Niroomand** provides a sufficient condition for a group to be finite.

Theorem (Pavrizi, Niroomand, 2012)

Let G be a **finitely generated** *group. Suppose that the non-abelian tensor square $[G, G^{\varphi}]$ is finite. Then G is finite.*

M. Parvizi, P. Niroomand,
On the structure of groups whose exterior or tensor square is a p-group, J. Algebra, **352** (2012), 347-353.

$$\text{If } [G, G^{\varphi}] \text{ is finite } \not\Longrightarrow \ G \text{ is a finite group.}$$

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Applications

Corollary

Let n be a positive integer. If G is a group such that

1. *the derived subgroup G' is finitely generated*
2. *the size of the conjugacy class $|\alpha^{\nu(G)}| \leq n$ for every $\alpha \in T_{\otimes}(G)$*

then G is a BFC-group.

Some remarks

Remark (1)

If we get rid of **hypothesis 1** *of course G is not a BFC-group.*

Example 1. Let p be a prime. We define the semi-direct product $G = A \rtimes C_2$, where $C_2 = \langle d \mid d^2 = 1 \rangle$, $A = C_{p^\infty}$ is the Prüfer group and

$$a^d = a^{-1},$$

for every $a \in A$.

Then we have:

- $G' = A$ is not finitely generated
- $|x^{\nu(G)}| \leq 4$ for every $x \in T_{\otimes}(G)$
- G is not a BFC-group

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Some remarks

Remark (2)

We cannot replace the **hypothesis 2** *by "*$|x^G| \leq n$ *for every* $x \in \Gamma(G)$*".*

Example 2. Let $G = \langle a, d \mid d^2 = 1, a^d = a^{-1} \rangle$ be the infinite dihedral group.

Then we have:

- $|x^G| \leq 2$ for every commutator $x \in \Gamma(G)$
- G' is an infinite subgroup of $\langle a \rangle$
- G is not a BFC-group

Some remarks

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Thank you for the attention!