Boundedly finite conjugacy classes of tensors

Carmine Monetta
University of Salerno

“Semigroups and Groups, Automata, Logics”
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This is a joint work\textsuperscript{1} with Raimundo Bastos.

\begin{itemize}
\item R. Bastos, C. Monetta, \textit{Boundedly finite conjugacy classes of tensors}, submitted.
\end{itemize}

\textsuperscript{1}Funded by GNSAGA
Let $G$ be a group. We denote by $G \otimes G$ the group generated by the symbols $g \otimes h$, with $g, h \in G$, which satisfy the following conditions:

- $gh \otimes x = (g^h \otimes x^h)(h \otimes x)$
- $g \otimes hx = (g \otimes x)(g^x \otimes h^x)$

for every $g, h, x \in G$.

This is called the **non-abelian tensor square of $G$**, and the generators $g \otimes h$ are called **tensors**.
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They showed that the third homotopy group of the suspension of an Eilenberg-MacLane space $K(G,1)$ satisfies

$$\pi_3 SK(G,1) \simeq \mu(G)$$

where $\mu(G)$ is the kernel of the derived map

$$k : G \otimes G \to G',$$

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In this presentation we will deal with some algebraic aspects related to the non-abelian tensor square.

More precisely, we will focus on the similarities between commutators and tensors.

Given a group $G$, let $G^\varphi$ be an isomorphic copy of $G$ ($g \mapsto g^\varphi$).

$$\nu(G) := \langle G, G^\varphi \mid [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, \; g_i \in G \rangle.$$  

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Non-abelian tensor square as a subgroup

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is the **commutator connection**:

**Proposition (Rocco, 1991)**

The map \( \Phi : G \otimes G \rightarrow [G, G^\varphi] \), defined by \( g \otimes h \mapsto [g, h^\varphi] \), for every \( g, h \in G \), is an **isomorphism**.

Since \([G, G^\varphi] \leq \nu(G)'\), to investigate properties of \( G \otimes G \), one can look at the commutators in the group \( \nu(G) \).
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The investigation from a **group theoretical point of view** started with a paper by Brown, Johnson, and Robertson.

R. Brown, D. L. Johnson, E. F. Robertson,

They compute the non-abelian tensor square of all non-abelian groups of order up to 30 using Tietze transformations.

However, this method is not appropriate for computing $G \otimes G$ for larger groups since we have $|G|^2$ generators and $2|G|^3$ relations.
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Independently, McDermott and Rocco obtained simplified presentations for $\nu(G)$ starting from some generating sequences of $G$ associated to some subnormal series.

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Donadze, Ladra and Thomas, proved that if a group $G$ belongs to some class of groups, then so $G \otimes G$ does.


They proved it for nilpotent by finite, solvable by finite, polycyclic by finite, nilpotent of nilpotency class $n$ and supersolvable groups.
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*On some closure properties of the non-abelian tensor product*, J. Algebra, **472** (2017), 399-413.

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**Remark**

Let \( \mathcal{X} \) be a class of groups “closed” under taking subgroups, direct products and central extensions of its elements.

If \( G \in \mathcal{X} \), then \( \nu(G) \in \mathcal{X} \).

One can consider the homomorphism \( \rho : \nu(G) \rightarrow G \) defined by \( g \mapsto g \) and \( g^\varphi \mapsto g \). The kernel of this map is denoted by \( \Theta(G) \). Then we have

\[
\frac{\nu(G)}{\Theta(G)} \cong G \quad \text{and} \quad \frac{\nu(G)}{[G, G^\varphi]} \cong G \times G
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and \( \mu(G) := [G, G^\varphi] \cap \Theta(G) \leq Z(\nu(G)) \), is a central subgroup.

Then \( \nu(G)/\mu(G) \) is isomorphic to a subgroup of \( G \times G \times G \). It follows that \( \nu(G) \in \mathcal{X} \).

**Examples of classes:** nilpotent, supersoluble, soluble, locally nilpotent, locally soluble, Engel, ...
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$$\Gamma(G) := \{[g, h] \mid g, h \in G\} \quad \text{and} \quad x^G := \{x^g \mid g \in G\}$$

A group $G$ is said to be a BFC-group if there exists a positive integer $n$ such that $|x^G| \leq n$ for every $x \in G$.

If $n$ is the least upper bound of the size of the conjugacy classes, then we say that $G$ is a $n$-BFC-group.

**Remark**

*If $|\Gamma(G)| = n$ is finite, then $G$ is a $k$-BFC-group for some $k \leq n$.***

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Elements with finitely many conjugates

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Theorem (Neumann, 1951)

A group $G$ is a BFC-group $\iff G'$ is finite $\iff \Gamma(G)$ is finite

B. H. Neumann,

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Wiegold obtained a quantitative version of the previous theorem.

Theorem (Wiegold, 1958)

If $G$ is a BFC-group with least upper bound of the conjugacy classes given by the positive integer $n$, then $|G'| \leq n^{\frac{1}{2}}n^4(\log_2 n)^3$.

J. Wiegold,

*Groups with boundedly finite classes of conjugate elements*, Proc. R. Soc.
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Conjecture and best bound

In the same paper, Wiegold conjecture for following bound.

**Conjecture (Wiegold 1958)**

If $G$ is an $n$-BFC-group, then $|G'| \leq n^{\frac{1}{2}}(1+\log_2 n)$.

The best bound known is due to Guralnick and Maróti.

**Theorem (Guralnick, Maróti, 2011)**

Let $G$ be an $n$-BFC-group with $n > 1$. Then $|G'| < n^{\frac{1}{2}}(7+\log_2 n)$.

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We denote by $T \otimes (G)$ the set of all tensors, i.e.,

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**Theorem (Bastos, Nakaoka, Rocco, 2018)**

The non-abelian tensor square $[G, G^\varphi]$ is finite if and only if $T \otimes (G)$ is finite.

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Let $n$ be a positive integer and assume that $|x^G| \leq n$ for every commutator $x \in \Gamma(G)$. Then the second derived subgroup $G''$ is finite with $n$-bounded order.

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Question

Le $n$ be a positive integer. Assume that $|\alpha^{\nu(G)}| \leq n$ for any $\alpha \in T \otimes (G)$. Is then $([G, G^\varphi])'$ finite?

Since $[G, G^\varphi] \leq \nu(G)'$, then $([G, G^\varphi])' \leq \nu(G)''$. 
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Question

Let $n$ be a positive integer. Assume that $|\alpha^{\nu(G)}| \leq n$ for any $\alpha \in T \otimes (G)$. Is then $([G, G^\varphi])'$ finite?

Since $[G, G^\varphi] \leq \nu(G)'$, then $([G, G^\varphi])' \leq \nu(G)'$.  

Non-abelian Tensor Square

Carmine MONETTA

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Theorem (Bastos, M., 2019)

Let $n$ be a positive integer. Suppose that $\left| \alpha^{\nu(G)} \right| \leq n$ for every $\alpha \in T_\otimes(G)$. Then $\nu(G)''$ is finite with $n$-bounded order.
If \([G, G^\varphi]\) is finite \(\implies G\) is a finite group.

For instance, the Prüfer group \(C_{p^\infty}\) is an example of an infinite group such that the non-abelian tensor square \([C_{p^\infty}, (C_{p^\infty})^\varphi]\) is trivial (and so, finite).

However, Parvizi and Niroomand provides a sufficient condition for a group to be finite.

**Theorem (Pavrizi, Niroomand, 2012)**

Let \(G\) be a **finitely generated** group. Suppose that the non-abelian tensor square \([G, G^\varphi]\) is finite. Then \(G\) is finite.

M. Parvizi, P. Niroomand,

*On the structure of groups whose exterior or tensor square is a \(p\)-group*, J. Algebra, **352** (2012), 347-353.
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Let $G$ be a **finitely generated** group. Suppose that the non-abelian tensor square $[G, G^\varphi]$ is finite. Then $G$ is finite.

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Corollary

Let \( n \) be a positive integer. If \( G \) is a group such that

1. the derived subgroup \( G' \) is finitely generated
2. the size of the conjugacy class \( |\alpha^{G'}| \leq n \) for every \( \alpha \in T_\otimes(G) \)

then \( G \) is a BFC-group.
Remark (1)

If we get rid of hypothesis 1 of course \( G \) is not a BFC-group.

Example 1. Let \( p \) be a prime. We define the semi-direct product \( G = A \rtimes C_2 \), where \( C_2 = \langle d \mid d^2 = 1 \rangle \), \( A = C_{p^{\infty}} \) is the Prüfer group and

\[
a^d = a^{-1},
\]

for every \( a \in A \).

Then we have:

- \( G' = A \) is not finitely generated
- \( |x^{\nu(G)}| \leq 4 \) for every \( x \in T_\otimes(G) \)
- \( G \) is not a BFC-group
Some remarks

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Example 1. Let $p$ be a prime. We define the semi-direct product $G = A \rtimes C_2$, where $C_2 = \langle d \mid d^2 = 1 \rangle$, $A = C_{p^\infty}$ is the Prüfer group and $a^d = a^{-1}$ for every $a \in A$.

Then we have:

- $G' = A$ is not finitely generated
- $|x^{\nu(G)}| \leq 4$ for every $x \in T_\otimes(G)$
- $G$ is not a BFC-group
Remark (2)

We cannot replace the hypothesis 2 by “\(|x^G| \leq n \text{ for every } x \in \Gamma(G)\)”.

Example 2. Let \( G = \langle a, d \mid d^2 = 1, a^d = a^{-1} \rangle \) be the infinite dihedral group.

Then we have:

- \(|x^G| \leq 2 \text{ for every commutator } x \in \Gamma(G)\)
- \(G'\) is an infinite subgroup of \(\langle a \rangle\)
- \(G\) is not a BFC-group
Remark (2)

*We cannot replace the hypothesis 2 by “$|x^G| \leq n$ for every $x \in \Gamma(G)$.”*

Example 2. Let $G = \langle a, d \mid d^2 = 1, a^d = a^{-1} \rangle$ be the infinite dihedral group.

Then we have:

- $|x^G| \leq 2$ for every commutator $x \in \Gamma(G)$
- $G'$ is an infinite subgroup of $\langle a \rangle$
- $G$ is not a BFC-group
Thank you for the attention!