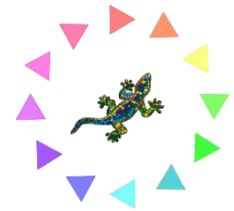


A Conjecture on Groups in which Normality is a Transitive Relation

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Historical Background

A group G is said to be a *T-group* if all subnormal subgroups of G are normal (i.e., if the normality in G is a transitive relation).

Clearly, every *Dedekind group* is a *T-group*, and the nilpotent *T-groups* are exactly the Dedekind ones. It follows that $\gamma_3(G) = \gamma_4(G)$, and that $C_G(G')$ is the *Fitting subgroup* of G , if G is a *T-group*.



- **Best, Taussky (1942)** *Finite groups with cyclic subgroups are T-groups.*
- **Zacher (1952)** *Characterization of finite soluble T-groups by means of Sylow tower properties.*
- **Gaschütz (1957)** *Let G be a **finite** soluble T-group and let $L = \gamma_3(G)$. Then L is abelian of odd order, G/L is a Dedekind group of order relatively prime to $|L|$. Moreover, G acts over L by conjugation as a group of power automorphisms.*



➤ **Robinson (1964)** *Description of the structure of infinite soluble T-groups (more complicated!)*

Some results:

- *Soluble T-groups are metabelian;*
- *Soluble T-groups are locally supersoluble;*
- *Finitely generated soluble T-groups are either finite or abelian.*



Clearly, the class T is quotient closed, but it is not subgroups closed since every simple group is a T -group.

A group G is called a \bar{T} -group if all subgroups of G are T -groups.

- ✓ *Every finite soluble T -group is a \bar{T} -group.*
- ✓ *All finite \bar{T} -groups are soluble.*



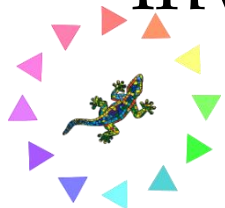
Theorem 1 (Robinson, 1964) *The soluble periodic \bar{T} -groups are precisely the periodic soluble T -groups with the property that $L = \gamma_3(G)$ has the 2-component $L_2 = 1$ and $\pi(L) \cap \pi(G/L) = \emptyset$.*

Theorem 2 (Robinson, 1964) *Let G be a non-periodic soluble \bar{T} -group. Then G is abelian.*



Some Examples

- ✓ A *Tarski p -group* (Ol'shanskii, 1979, i.e., an infinite (simple) group in which all proper non-trivial subgroups have order p), satisfies the \bar{T} -property.
- ✓ The locally dihedral 2-group D_{2^∞} is a soluble T -group which does not satisfy the property \bar{T} .
- ✓ Let p be an odd prime. Then the semidirect product $G = \langle x \rangle \rtimes Z(p^\infty)$, where x is the inversion, is a soluble \bar{T} -group.



In general, a locally \bar{T} -group need **not split** over the last term of its lower central series.

- ✓ **Kovacs, Neumann and De Vries (1961)**, constructed a locally finite group G such that the last term L of its lower central series is a countable quasi-central abelian subgroup such that $L_2=1$ and G/L is an uncountable elementary abelian 2-group. Moreover, each Sylow 2-subgroup of G is countable and hence it cannot be a complement of L in G .



Locally Graded Groups

(Chernikov, 1970) A group G is said to be *locally graded* (\mathcal{LG}) if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image.



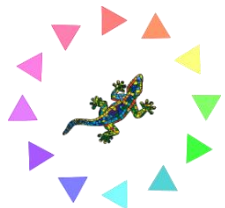
Some Subclasses of $\mathcal{L}\mathcal{G}$

- ✓ Locally finite groups.
- ✓ Locally soluble groups.
- ✓ (Weakly) Radical groups.
- ✓ Groups with no infinite simple sections.
- ✓ Residually finite groups.
- ✓ Free groups (Iwasawa).



Remarks

- \mathcal{LG} is the most natural class of generalized soluble groups which is useful if one wishes to avoid the presence of finitely generated infinite simple groups (Tarski groups) or other similar pathologies.
- Some situations become trivial if they are considered in \mathcal{LG} . For example, *there is no infinite non-abelian group all of whose proper subgroups are abelian.*



- Some problems which have negative answer in general, have an affirmative solution for $\mathcal{L}\mathcal{G}$ -groups. An example is the **Bounded Burnside Problem** which, in general, has negative answer (Novikov and Adian, 1964).

*Let G be a locally graded group of finite exponent.
Then G is locally finite.*

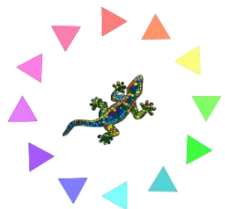
(The statement follows immediately from the positive answer to the **Restricted Burnside Problem** – Zelmanov, 1989;

Fields Medal in 1994 for this work)



Locally Graded \bar{T} -groups

Theorem 3 *Let G be a periodic locally graded \bar{T} -group. Then G is soluble (and hence G is metabelian).*

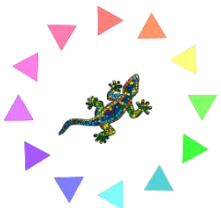


Conjecture

Let G be a non-periodic locally graded \bar{T} -group. Then G is abelian.

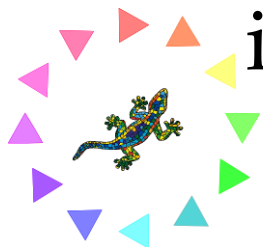
(Khukhro, Sozutov, **The Kourovka Notebook**, vol. 16, (2006), Question 14.36 (de Giovanni)).

(Ol'shanskii, 1979) *There exists a torsion free non-abelian simple group in which all proper non-trivial subgroups are cyclic.*



Proposition 4 *Let G be a non-periodic \bar{T} -group having no infinite simple sections. Then G is abelian.*

Proof – For a contradiction, assume that G contains a finitely generated subgroup E which is not metabelian. Since $E/E^{(2)}$ is polycyclic, it follows that $E^{(2)}$ is the normal closure of a finite subset, and hence by Zorn's lemma it contains a maximal E -invariant subgroup M .



Since E is a T -group, then $E^{(2)}/M$ is simple, and so it is a finite \bar{T} -group by hypothesis.

Therefore, E/M is soluble, and hence $E^{(2)} \leq M$, a contradiction. \square

Remark - Clearly, the class \mathcal{LG} of locally graded groups is closed under subgroups and extensions. But, as any free group is locally graded, then \mathcal{LG} is not quotient closed. On the other hand, it is useful to recall the following result:

(Longobardi, Maj, Smith, 1995) *Let G be any locally graded group. If N is a normal **soluble** subgroup of G , then G/N is locally graded.*



Lemma 5 *Let G be a non-periodic locally graded \bar{T} -group. Then the set L consisting of elements of finite order of G is a subgroup such that $L \leq Z(G)$.*

Proof - Let x and y be elements of finite order of G . Assume that the subgroup $\langle x, y \rangle$ is infinite. As G is locally graded, it can be constructed a strictly descending sequence $(H_n)_{n \in \mathbb{N}}$ of normal subgroups of G



$$H = H_1 > H_2 > \dots > H_n > \dots$$

such that H/H_n is finite (and even metabelian) for all n . Put $X = \bigcap_{n \in \mathbb{N}} H_n$. Then the factor group H/X is an infinite finitely generated soluble \bar{T} -group, and hence it is abelian. Thus H/H' is infinite, a contradiction. It follows that $\langle x, y \rangle$ is finite. Therefore L is a subgroup of G . Finally, let $x \in L$ and $y \in G \setminus L$ and put $K = \langle x, y \rangle$.



Clearly, $K = \langle y \rangle \rtimes L_1$, where L_1 is the torsion subgroup of K . By Theorem 3 K is soluble, and hence it is abelian. Thus x commutes with every element of $G \setminus L$, and hence $x \in Z(G)$. \square



Embeddings Properties and Classes T and \bar{T}

Let \mathfrak{U} be an universe of groups. Denote by $\mathfrak{L}(G)$ the lattice subgroup of a group G . A *subgroup embedding property* (related to \mathfrak{U}) is a map f which associates with each group G in \mathfrak{U} a subset $f(G)$ of $\mathfrak{L}(G)$ and satisfies the equation $\alpha(f(G))=f(\alpha(G))$, for all group isomorphisms $\alpha: G \rightarrow \alpha(G)$.

Obvious example: (sub)normality.



❖ Subnormalizer Condition (Misovskikh, 2002)

A subgroup X of a group G satisfies the *subnormalizer condition* if for every subgroup Y of G such that

$$X \leq Y \leq N_G(X),$$

then

$$N_G(Y) \leq N_G(X).$$



Different notations:

- *Pseudonormal* (de Giovanni, Vincenzi, 2003)
- *Transitively normal* (Kurdachenko, Subbotin, 2006)

Proposition 6 *A group G has the \bar{T} -property if and only if all subgroups of G satisfy the subnormalizer condition.*



Subnormalizer Condition

Weakly Normality

Pronormality



❖ Pronormality (P. Hall, Rose, ~ 1967)

A subgroup X of a group G is said to be *pronormal* if the X and X^g are conjugate in the subgroup $\langle X, X^g \rangle$, for all $g \in G$.

Clearly, every pronormal subgroup satisfies the subnormalizer condition. Moreover, *if X is a subgroup of a group G such that X is pronormal and subnormal, then X is normal.*



Some Examples

- ✓ Normal subgroup of arbitrary group.
- ✓ Maximal subgroup of arbitrary group.
- ✓ Sylow subgroup of finite groups.
- ✓ Hall subgroups of finite soluble groups.
- ✓ Carter subgroups of finite soluble groups.



Theorem 7 (Peng, 1969) *Let G be a finite group.*

Then the following are equivalent:

- 1. Every subgroup of G is pronormal;*
- 2. G is a \bar{T} -group.*

In the **infinite** case the strongest result about the periodic groups with their subgroups pronormal is due to Kuzennyi and Subbotin (1987).

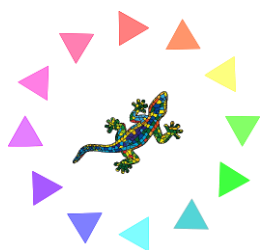


Theorem 8 (Kuzennyi, Subbotin, 1987)

A periodic locally graded group has all its subgroups pronormal if and only if it is isomorphic to a group G having the following structure:

- 1. G is periodic and contains a normal abelian subgroup A such that G/A is a Dedekind group;*
- 2. $\pi(A) \cap \pi(G/A) = \emptyset$;*
- 3. Elements of G induce power automorphisms in A and $A = [A, G]$;*
- 4. If Q is a Sylow $\pi(G/A)$ -subgroup of G , then*

$$G = Q \rtimes A.$$



Remarks

- There exists a locally finite \bar{T} -group G which contains non-pronormal subgroups (Kuzennyi, Subbotin, 1987). Note that G splits over the last term of its central lower series.
- The quoted example of Kovacs, Neumann and De Vries is a locally finite \bar{T} -group G which does **not** splits over the last term of its central lower series. It follows that every 2-Sylow subgroup of G is not pronormal.



Theorem 9 (Robinson, A. R., Vincenzi, 2007)

Let G be a non-periodic locally graded group, all of whose subgroups are pronormal. Then G is abelian.

(Key) Lemma 10 (Robinson, A. R., Vincenzi, 2007)

Let G be a finitely generated locally graded group with finite upper rank. If G is HNN-free, then G is polycyclic-by-finite.



❖ Weakly normality (Müller, 1966)

A subgroup X of a group G is said to be *weakly normal* if $X^g = X$ whenever g is an element of G such that $X^g \leq N_G(X)$.

Every pronormal subgroup is weakly normal (vice-versa is **not** true!). Moreover,

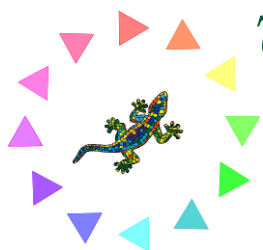
if X is a subgroup of a group G such that X is weakly normal and subnormal, then X is normal.



Theorem 11 (Ballester-Bolinches, Esteban-Romero, 2003) *Let G be a finite group. Then G is a soluble T -group if and only if every subgroup of G is weakly normal.*

Theorem 12 (A. R., 2012) *Let G be a locally graded group. Then the following hold:*

- 1. If G is periodic, then G is a \bar{T} -group if and only if every subgroup of G is weakly normal.*
- 2. If G is non-periodic and all its subgroups are weakly normal, then G is abelian.*



Recent Results

A group G is said to be a T_c -group (a \bar{T}_c -group) if every cyclic subnormal subgroup of G is normal (every subgroup of G is a T_c -group) (Xu, 1988; Sakamoto, 2001; Robinson, 2005)

Theorem 13 (Esteban-Romero, de Giovanni, A. R., 2019) *Let G be a torsion free weakly radical group with the \bar{T}_c -property. Then G is abelian.*



Remarks

- ✓ The infinite dihedral group D_∞ is a \overline{T}_c -group
So there exists (finitely generated) soluble non-periodic \overline{T}_c -groups which are not abelian.
- ✓ The consideration of D_∞ also shows that the class \overline{T}_c is non quotient closed.



❖ Weak subnormalizer condition

(Esteban-Romero, de Giovanni, A. R., 2019)

A subgroup X of a group G satisfies the *weak subnormalizer condition* if for every **non-normal** subgroup Y of G such that

$$X \leq Y \leq N_G(X),$$

then

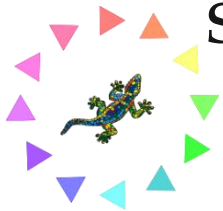
$$N_G(Y) \leq N_G(X).$$



Theorem 13 (Esteban-Romero, de Giovanni, A. R., 2019)

Let G be a weakly radical group all of whose subgroups satisfy the weak subnormalizer condition. Then the commutator subgroup G' of G is periodic. In particular, G is abelian, if it is torsion-free.

The consideration of the direct product $\mathbb{Z} \times S_3$ shows that there exist non-periodic soluble non-abelian groups all of whose subgroups satisfy the weak subnormalizer condition.



Many Thanks