A Conjecture on Groups in which Normality is a Transitive Relation

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Historical Background

A group G is said to be a *T-group* if all subnormal subgroups of G are normal (i.e., if the normality in G is a transitive relation). Clearly, every *Dedekind group* is a T-group, and the nilpotent T-groups are exactly the Dedekind ones. It follows that $\gamma_3(G)=\gamma_4(G)$, and that $C_G(G')$ is the *Fitting subgroup* of G, if G is a T-group.

- Best, Taussky (1942) *Finite groups with cyclic subgroups are T-groups.*
- Zacher (1952) *Characterization of finite soluble T-groups by means of Sylow tower properties.*
- Gaschütz (1957) *Let G be a finite soluble T-group and let L= γ_3(G). Then L is abelian of odd order, G/L is a Dedekind group of order relatively prime to |L|. Moreover, G acts over L by conjugation as a group of power automorphisms.*

- Robinson (1964) *Description of the structure of infinite soluble T-groups (more complicated!)*

Some results:

- *Soluble T-groups are metabelian;*
- *Soluble T-groups are locally supersoluble;*
- *Finitely generated soluble T-groups are either finite or abelian.*

Clearly, the class T is quotient closed, but it is not subgroups closed since every simple group is a T-group.

A group G is called a $\bar{T}$-*group* if all subgroups of G are T-groups.

- ✓ *Every finite soluble T-group is a* $\bar{T}$*-group.*
- ✓ *All finite* $\bar{T}$*-groups are soluble.*

Theorem 1 (Robinson, 1964) *The soluble periodic $\bar{T}$-groups are precisely the periodic soluble T-groups with the property that $L=\gamma_3(G)$ has the 2-component $L_2=1$ and $\pi(L) \cap \pi(G/L)=\emptyset$.*

Theorem 2 (Robinson, 1964) *Let G be a non-periodic soluble $\bar{T}$-group. Then G is abelian.*

Some Examples

- A *Tarski p-group* (**Ol'shanskii**, 1979, i.e., an infinite (simple) group in which all proper non-trivial subgroups have order *p*), satisfies the $\bar{T}$-property.
- The locally dihedral 2-group D_{2^∞} is a soluble T-group which does not satisfy the property $\bar{T}$.
- Let p be an odd prime. Then the semidirect product $G = < x > \ltimes Z(p^\infty)$, where x is the inversion, is a soluble $\bar{T}$-group.

In general, a locally $\bar{T}$-group need not split over the last term of its lower central series.

✓ Kovacs, Neumann and De Vries (1961), constructed a locally finite group G such that the last term L of its lower central series is a countable quasi-central abelian subgroup such that $L_2=1$ and G/L is an uncountable elementary abelian 2-group. Moreover, each Sylow 2-subgroup of G is countable and hence it cannot be a complement of L in G.

Locally Graded Groups

(Chernikov, 1970) A group G is said to be *locally graded* ($\mathcal{LG}$) if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image.

Some Suclasses of $\mathcal{LG}$

- ✓ Locally finite groups.
- ✓ Locally soluble groups.
- ✓ (Weakly) Radical groups.
- ✓ Groups with no infinite simple sections.
- ✓ Residually finite groups.
- ✓ Free groups (Iwasawa).

Remarks

- $\mathcal{LG}$ is the most natural class of generalized soluble groups which is useful if one wishes to avoid the presence of finitely genereted infinite simple groups (Tarski groups) or other similar patologies.
- Some situations become trivial if they are considered in $\mathcal{LG}$. For example, *there is no infnite non-abelian group all of whose proper subgroups are abelian.*

- Some problems which have negative answer in general, have an affirmative solution for $\mathcal{LG}$-groups. An example is the Bounded Burnside Problem which, in general, has negative answer (Novikov and Adian, 1964).

Let G be a locally graded group of finite exponent.

Then G is locally finite.

(The statement follows immediately from the positive answer to the Restricted Burnside Problem – **Zelmanov**, 1989;

Fields Medal in **1994** for this work)

Locally Graded $\bar{T}$-groups

Theorem 3 *Let G be a periodic locally graded $\bar{T}$-group. Then G is soluble (and hence G is metabelian).*

Conjecture

Let G be a non-periodic locally graded $\overline{T}$-group. Then G is abelian.

(Khukhro, Sozutov, **The Kourovka Notebook**, vol. 16, (2006), Question 14.36 (de Giovanni)).

(**Ol'shanskii**, 1979) *There exists a torsion free non-abelian simple group in which all proper non-trivial subgroups are cyclic.*

Proposition 4 *Let G be a non-periodic $\bar{T}$-group having no infinite simple sections. Then G is abelian.*

Proof – For a contradicion, assume that G contains a finitely generated subgroup E subgroup E which is not metabelian. Since $E/E^{(2)}$ is polycyclic, it follows that $E^{(2)}$ is the normal closure of a finite subset, and hence by Zorn's lemma it contains a maximal E-invariant subgroup M.

Since E is a T-group, then $E^{(2)}/M$ is simple, and so it a finite $\bar{T}$-group by hypothesis. Therefore, E/M is soluble, and hence $E^{(2)} \leq M$, a contradiction. $\square$

Remark – Clearly, the class $\mathcal{LG}$ of locally graded groups is closed under subgroups and extensions. But, as any free group is locally graded, then $\mathcal{LG}$ is not quotient closed. On the other hand, it is useful to recall the following result:

(Longobardi, Maj, Smith, 1995) *Let G be any locally graded group. If N is a normal **soluble** subgroup of G, then G/N is locally graded.*

Lemma 5 *Let G be a non-periodic locally graded $\bar{T}$-group. Then the set L consisting of elements of finite order of G is a subgroup such that $L \leq Z(G)$.*

Proof – Let x and y be elements of finite order of G. Assume that the subgroup $<x,y>$ is infinite. As G is locally graded, it can be constructed a strictly descending sequence $(H_n)_{n \in N}$ of normal subgroups of G

$$H=H_1>H_2>\ldots>H_n>\ldots$$

such that H/H_n is finite (and even metabelian) for all n. Put $X = \bigcap_{n\in N} H_n$. Then the factor group H/X is an infinite finitely generated soluble $\bar{T}$-group, and hence it is abelian. Thus H/H' is infinite, a contradiction. It follows that $\langle x,y\rangle$ is finite. Therefore L is a subgroup of G. Finally, let $x\in L$ and $y\in G\setminus L$ and put $K=\langle x,y\rangle$.

Clearly, $K = \langle y \rangle \ltimes L_1$, where L_1 is the torsion subgroup of K. By Theorem 3 K is soluble, and hence it is abelian. Thus x commute with every element of $G \setminus L$, and hence $x \in Z(G)$. $\square$

Embeddings Properties and Classes T and $\overline{T}$

Let $\mathfrak{U}$ be an universe of groups. Denote by $\mathfrak{L}(G)$ the lattice subgroup of a group G. A *subgroup embedding property* (related to $\mathfrak{U}$) is a map f which associates with each group G in $\mathfrak{U}$ a subset $f(G)$ of $\mathfrak{L}(G)$ and satisfies the equation $\alpha(f(G))=f(\alpha(G))$, for all group isomorphisms $\alpha: G \to \alpha(G)$.

Obvious example: (sub)normality.

❖ Subnormalizer Condition (Misovskikh, 2002)

A subgroup X of a group G satisfies the *subnormalizer condition* if for every subgroup Y of G such that

$$X \leq Y \leq N_G(X),$$

then

$$N_G(Y) \leq N_G(X).$$

Different notations:

- *Pseudonormal* (de Giovanni, Vincenzi, 2003)
- *Transitively normal* (Kurdachenko, Subbotin, 2006)

Proposition 6 *A group G has the $\bar{T}$-property if and only if all subgroups of G satisfy the subnormalizer condition.*

Subnormalizer Condition

↑

Weakly Normality

↑

Pronormality

❖ Pronormality (P. Hall, Rose, ~ 1967)

A subgroup X of a group G is said to be *pronormal* if the X and X^g are conjugate in the subgroup $< X, X^g>$, for all $g \in G$.

Clearly, every pronormal subgroup satisfies the subnormalizer condition. Moreover, *if X is a subgroup of a group G such that X is pronormal and subnormal, then X is normal.*

Some Examples

- ✓ Normal subgroup of arbitrary group.
- ✓ Maximal subgroup of arbitrary group.
- ✓ Sylow subgroup of finite groups.
- ✓ Hall subgroups of finite soluble groups.
- ✓ Carter subgroups of finite soluble groups.

Theorem 7 (Peng, 1969) *Let G be a finite group. Then the following are equivalent:*

1. Every subgroup of G is pronormal;

2. G is a $\bar{T}$-group.

In the infinite case the strongest result about the periodic groups with their subgroups pronormal is due to Kuzennyi and Subbotin (1987).

Theorem 8 (Kuzennyi, Subbotin, 1987)

A periodic locally graded group has all its subgroups pronormal if and only if it is isomprphic to a group G having the following structure:

1. *G is periodic and contains a normal abelian subroup A such that G/A is a Dedekind group;*
2. $\pi(A)\cap\pi(G/A)=\emptyset$;
3. *Elements of G induce power automorphisms in A and $A=[A,G]$;*
4. *If Q is a Sylow $\pi(G/A)$-subgroup of G, then*

$$G = Q \ltimes A.$$

Remarks

- There exists a locally finite $\bar{T}$-group G which contains non-pronormal subgroups (Kuzennyi, Subbotin, 1987). Note that G **splits** over the last term of its central lower series.
- The quoted example of Kovacs, Neumann and De Vries is a locally finite $\bar{T}$-group G which does **not splits** over the last term of its central lower series. It follows that every 2-Sylow subgroup of G is not pronormal.

Theorem 9 (Robinson, A. R., Vincenzi, 2007)

Let G be a non-periodic locally graded group, all of whose subgroups are pronormal. Then G is abelian.

(Key) **Lemma 10 (Robinson, A. R., Vincenzi, 2007)**

Let G be a finitely generated locally graded group with finite upper rank. If G is HNN-free, then G is polycyclic-by-finite.

Weakly normality (Müller, 1966)

A subgroup X of a group G is said to be *weakly normal* if $X^g = X$ whenever g is an element of G such that $X^g \leq N_G(X)$.

Every pronormal subgroup is weakly normal (vice-versa is **not** true!). Moreover,

if X is a subgroup of a group G such that X is weakly normal and subnormal, then X is normal.

Theorem 11 (Ballester-Bolinches, Esteban-Romero, 2003) *Let G be a finite group. Then G is a soluble T-group if and only if every subgroup of G is weakly normal.*

Theorem 12 (A. R., 2012) *Let G be a locally graded group. Then the following hold:*

1. *If G is periodic, then G is a $\bar{T}$-group if and only if every subgroup of G is weakly normal.*
2. *If G is non-periodic and all its subgroups are weakly normal, then G is abelian.*

Recent Results

A group G is said to be a *T_c-group* (a *$\overline{T}_c$-group*) if every cyclic subnormal subgroup of G is normal (every subgroup of G is a T_c-group) **(Xu, 1988; Sakamoto, 2001; Robinson, 2005)**

Theorem 13 (Esteban-Romero, de Giovanni, A. R., 2019) *Let G be a torsion free weakly radical group with the $\overline{T}_c$-property. Then G is abelian.*

Remarks

- ✓ The infinite dihedral group D_∞ is a $\overline{T}_c$-group So there exists (finitely genereted) soluble non-periodic $\overline{T}_c$-groups which are not abelian.

- ✓ The consideration of D_∞ also shows that the class $\overline{T}_c$ is non quotient closed.

❖ **Weak subnormalizer condition**
(Esteban-Romero, de Giovanni, A. R., 2019)

A subgroup X of a group G satisfies the *weak subnormalizer condition* if for every **non-normal** subgroup Y of G such that

$$X \leq Y \leq N_G(X),$$

then

$$N_G(Y) \leq N_G(X).$$

Theorem 13 (Esteban-Romero, de Giovanni, A. R., 2019)

Let G be a weakly radical group all of whose subgroups satisfy the weak subnormalizer condition. Then the commutator subgroup G' of G is periodic. In particular, G is abelian, if it is torsion-free.

The consideration of the direct product $\mathbb{Z} \times S_3$ shows that there exist non-periodic soluble non-abelian groups all of whose subgroups satisfy the weak subnormalizer condition.

Many Thanks