Isoclinism and classification of finite p-groups

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SandGAL - Semigroups and Groups, Automata, Logics
AUGURI SANDRA!

P. Hall, *The classification of prime-power groups*, Journal für die reine und angewandte Mathematik, 1940.

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- Clearly an equivalence relation among groups, coarser than isomorphism.
- Finite p-groups of odd order from now on.

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- In fact, since all the elements of a coset of $Z(G)$ have the same centralizer, the set of the elements of G whose conjugacy class has length p^i is a union of say n_i cosets of $Z(G)$. We have that $q_i = n_i|Z(G)|/p^i$. Dividing by $|G|$ we have the theorem: $q_i/|G| = n_i/|G : Z(G)|p^i$, as n_i is clearly an isoclinic invariant, and so is $|G : Z(G)|$.

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- As a corollary, the set of the conjugacy classes lengths, and in particular the *breadth* $b(G)$ (the largest i for which $q_i \neq 0$) and the *commutativity* $k(G)/|G|$ are isoclinic invariants ($k(G)$ is the sum of the q_i's, the usual class number).

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- As a consequence, the set of the degrees of the characters, and in particular the largest character degree, are isoclinic invariants.
- P. Hall then proves that in each "family" (i.e. equivalence class under the relation of isoclinism) there are groups for which $Z(G) \leq G'$, called *stem groups* of the family, and that they are all of the same minimal order among the groups of the family.

- Stem p-groups of breadth 1 (i.e. the non-central classes have order p) are the extraspecial p-groups. (Knoche 1951, Burnside 1911)

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- A bound on $b(G)$ is not enough to bound the order of the corresponding stem groups.

- p-groups of small breadth have been studied by many authors: e.g. Gavioli, Mann, Monti, Previtali and S. (1998), and, with different techniques, Parmeggiani and Stellmacher (1999) gave a classification of p-groups of breadth 2 and 3.

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- The set of possible values for the commutativity, i.e. the isoclinic invariant $k(G)/|G|$ for such groups, is surprisingly small.

- A systematic study was started, to understand how strictly the commutativity of the group depends on its structure.

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- We studied the behavior of the breadth polynomial under taking maximal subgroups, maximal quotients, and direct products.

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- Among them we find the wreath product of a cyclic p-group with a group of order p: stem groups of unbounded order and unbounded breadth.
- A bound on the largest character degree is not enough to bound the order of the corresponding stem groups.
- p-groups with bounded largest character degree have been extensively studied, see e.g. Chapter 12 of Isaacs book in Character Theory.

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- We defined the *breadth-degree type* of a finite p-group G as the pair$(b(G), d(G))$, where $p^{d(G)}$ is the largest irreducible character degree of G, and we classified easily the groups of small breadth-degree type.

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- We constructed examples showing that a bound on $b(G)$ or on $d(G)$ does not yield a bound on $|G|$, even if we assume that G is a stem group.

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- Only a finite number of stem p-groups for each choice of b and d.

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- By the result of Isaacs, the stem p-groups of class 2 are special.

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- T.K. Naik, R.D. Kitture and M.K.Yadav have shown that, up to isoclinism, these are the only examples of stem p-groups of class 3 whose non-central conjugacy classes have the same length.
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- They prove that $G/Z(G)$ is isomorphic to $U_3(n)$ and that G' and $H(n)'$ are isomorphic.
- Finally they show the uniqueness of the commutation map.

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- Theorem (Chermak-Delgado): $\mathcal{F}_1(G)$ is a sublattice of the lattice of all subgroups of G, as products and intersections of elements of $\mathcal{F}_1$ are in $\mathcal{F}_1$. See Isaacs book on finite groups.

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- V. Russo, A Morresi-Zuccari and S. (2018) observed that many properties of $\mathcal{F}_1(G)$ are invariant under isoclinism, and applied this to the study of its structure for groups in which the only element of $\mathcal{F}_1(G)$ that contains its centralizer is G itself (SMCL groups).